APTS ASP Simple Exercises 3

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May 2011

1. Suppose that N is a Poisson process of rate α. Working with the result

$$\mathbb{P}\left[N_t = k\right] \quad = \quad \frac{(\alpha t)^k}{k!} e^{-\alpha t} \qquad \text{for } k = 0, 1, 2, \ldots\,,$$

   show that

   (i) $\frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathbb{P}\left[N_t > 0\right] \to \alpha$ as $t \to 0$,

   (ii) $\frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathbb{P}\left[N_t > 1\right] \to 0$ as $t \to 0$,

   (iii) $\frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathbb{P}\left[N_t = 0\right] \to -\alpha$ as $t \to 0$.

2. In the context of question 1, show that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathbb{P}\left[N_t > k\right] \quad = \quad \alpha \times \frac{(\alpha t)^k}{k!} e^{-\alpha t} \qquad \text{for } k = 0, 1, 2, \ldots\,.$$

   Hence deduce that the time to the k^{th} incident has a Gamma distribution, and write down the Gamma distribution parameters.

3. In the context of question 1, show that

$$X_t \quad = \quad N_t - \alpha t$$

   determines a martingale.

4. In the context of question 1, show that

$$Y_t \quad = \quad X_t^2 - \alpha t$$

   determines a martingale (X given as in question 3).

5. Suppose now that X is merely a nonnegative random variable, and h is an integrable function on $[0, t]$. Show that

$$\mathbb{E}\left[\int_0^{\min\{t,X\}} h(u)\,\mathrm{d}\,u\right] \quad = \quad \int_0^t \mathbb{P}\left[X > u\right] h(u)\,\mathrm{d}\,u\,.$$

6. with X as in question 5, suppose that

$$\mathbb{P}\left[X > t\right] \quad = \quad \exp\left(-\int_0^t h(s)\,\mathrm{d}\,s\right)\,.$$

   Show that

$$\mathbb{I}_{[X \le t]} - \int_0^{\min\{t,X\}} h(u)\,\mathrm{d}\,u$$

   determines a martingale.

7. Suppose that $X_1, X_2, \ldots$ are independent mean-zero unit-variance random variables, such that for some constant C we have $\mathbb{E}\left[|X_i|^3\right] < C$ for all i. Show that the sequence $X_1, X_2, \ldots$ satisfies the Lindeberg condition (so that $(X_1 + \ldots + X_n)/\sqrt{\mathrm{Var}\left[X_1 + \ldots + X_n\right]}$ tends to normality). (HINT: $\mathbb{E}\left[X_i^2; X_i^2 > \varepsilon^2 n\right] \leq \frac{1}{\varepsilon\sqrt{n}}\,\mathbb{E}\left[|X_i|^3\right]$.)

8. Show that in general the Lyapunov condition implies the Lindeberg condition.