APTS ASP Simple Exercises 4

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1. Recall that the total variation distance between two probability distributions μ and ν on $\mathcal{X}$ is given by

$$\text{dist}_{\text{TV}}(\mu, \nu) \;=\; \sup_{A \subseteq \mathcal{X}} \{\mu(A) - \nu(A)\} \,.$$

   Show that this is equivalent to the distance

$$\sup_{A \subseteq \mathcal{X}} |\mu(A) - \nu(A)| \,.$$

2. Show that if $\mathcal{X}$ is discrete, then

$$\text{dist}_{\text{TV}}(\mu, \nu) \quad = \quad \tfrac{1}{2} \sum_{y \in \text{X}} |\mu(y) - \nu(y)| \,.$$

   (Here we *do* need to use the absolute value on the RHS!)
   Hint: consider $A = \{y : \mu(y) > \nu(y)\}$.

3. Suppose now that μ and ν are density functions on $\mathbb{R}$. Show that

$$\text{dist}_{\text{TV}}(\mu, \nu) = 1 - \int_{-\infty}^{\infty} \min\{\mu(y), \nu(y)\} dy \,.$$

   Hint: remember that $|\mu - \nu| = \mu + \nu - 2\min\{\mu, \nu\}$.

4. Let X be a random walk on $\mathbb{R}$, with increments given by the standard normal distribution. Show that any bounded set is small of lag 1. Does there exist $k \geq 1$ such that *the whole state space* is small of lag k?

5. Consider a Vervaat perpetuity X, where

$$X_0 = 0; \qquad X_{n+1} = U_{n+1}(X_n + 1) \,,$$

   and where $U_1, U_2, \ldots$ are independent Uniform$(0, 1)$. Find a small set for this chain.

6. Recall the regeneration idea from Section 7.1: suppose that C is a small set (with lag 1) for a ϕ-recurrent chain X, *i.e.*

$$\mathbb{P}\left[X_1 \in A | X_0 = x \in C\right] \quad \geq \quad \alpha \nu(A) \,,$$

   and that $X_n \in C$. Then with probability α let $X_{n+1} \sim \nu$, and otherwise let it have transition distribution $\frac{p(x,\cdot) - \alpha\nu(\cdot)}{1-\alpha}$.

(a) Check that this latter expression really is a probability distribution!

(b) Check that X_{n+1} constructed in this manner obeys the correct transition distribution from X_n.

7. Define a reflected random walk as follows: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$ with independent Z_{n+1} of continuous density $f(z)$, with

$$\mathbb{E}[Z_{n+1}] < 0 \quad \text{and} \quad \mathbb{P}[Z_{n+1} > 0] > 0\,.$$

Show that the Foster-Lyapunov criterion for positive recurrence holds, using $\Lambda(x) = x$.

8. Reflected Simple Asymmetric Random Walk: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$ with independent Z_{n+1} such that

$$\mathbb{P}[Z_{n+1} = -1] = 1 - \mathbb{P}[Z_{n+1} = +1] > \frac{1}{2}\,.$$

Show that the Foster-Lyapunov criterion for geometric ergodicity holds, using $\Lambda(x) = e^{ax}$ for small positive a.