APTS Applied Stochastic Processes, Oxford, March 2023
Exercise Sheet for Assessment

The work here is "light-touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question. It may be helpful to ensure you are using a version of the notes put on the web *after* the APTS week concluded.

1 Continuous-time Markov chains, queuing and reversibility

Customers join a supermarket queue at constant rate $\lambda > 0$. The person serving the queue does not like the queue to get too long, and therefore serves customers at rate μx, where x is the number of customers currently in the queue and $\mu > 0$ is a constant. Let X_t be the number of customers in the queue at time t.

(a) Write down the transition rates $q_{x,y}$ for the continuous-time Markov chain $(X_t)_{t\geq 0}$.

(b) Find the stationary distribution for $(X_t)_{t\geq 0}$. Do you need any conditions on λ and μ?

(c) If an observer stands outside the supermarket after a long time has passed, at what rate do they see customers leave the shop?

2 Martingales and optional stopping

Consider a population of K voters, where each voter has one of two possible opinions representing their preference for one of two parties: **Red** (voter preference for red party), or **Blue** (voter preference for blue party).

Suppose that at time 0 there are $R \geq 0$ voters with the opinion **Red**, and $K - R \geq 0$ voters with the opinion **Blue**. At each time $n = 1, 2, \ldots$, the opinion of the voters is randomly updated as follows. Choose two voters u and v independently and uniformly at random (the possibility $u = v$ being allowed) and update the opinion of u to agree with that of v (so if u and v have the same opinion, or if $u = v$, then nothing changes).

Let T be the time that the population reaches a *consensus*, i.e., the first time that all voter opinions are the same. Let $Y = (Y_0, Y_1, \ldots)$ be the Markov chain on $\{0, 1, 2, \ldots, K\}$, where Y_n counts the number of voters whose opinion is **Red** at time n.

(a) Show that the Markov chain Y has transition probabilities:

$$p_{x,y} = \begin{cases} \dfrac{x(K-x)}{K^2} & \text{if } y = x+1 \text{ or } y = x-1, \\[2ex] 1 - \dfrac{2x(K-x)}{K^2} & \text{if } y = x, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

(b) Show that Y is a martingale, confirm that T is a stopping time, and use this to calculate the probability that all voters have common opinion **Red** at time T.

Now, let $S = (S_n, n \geq 0)$ be a *lazy* symmetric simple random walk on $\mathbb{Z}$, started at $R \in \{0, 1, \ldots, K\}$. That is, $S_0 = R$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$, where $X_1, X_2, \ldots$ are independent and identically distributed random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/4$ and $\mathbb{P}(X_1 = 0) = 1/2$. Let $\widetilde{T} := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = K\}$.

(c) State the distribution of $S_{\widetilde{T}}$, by comparison with part (b) or otherwise. Is $\mathbb{E}(T)$ larger or smaller than $\mathbb{E}(\widetilde{T})$? Justify your answers with a brief explanation.

(d) Show that $M_n = (S_n)^2 - n/2$ is a martingale, and calculate $\mathbb{E}(\widetilde{T})$.

3 Detailed balance and small sets

Suppose we have a density π on $\mathbb{R}$, which is continuous and strictly positive on the whole of $\mathbb{R}$. Consider a Metropolis-Hastings chain X for sampling from π. If $X_n = x \in \mathbb{R}$ then a new state y for X_{n+1} is proposed using density $\kappa(x, y)$, and then accepted with probability

$$\alpha(x, y) = \min\left\{\frac{\pi(y)\kappa(y, x)}{\pi(x)\kappa(x, y)}, 1\right\}.$$

Thus transitions of X take place according to the density

$$p(x, y) = \kappa(x, y)\alpha(x, y), \qquad y \neq x,$$

and with probability of remaining at the same point given by

$$\mathbb{P}(X_{n+1} = x \mid X_n = x) = \int_{\mathbb{R}} \kappa(x, y)(1 - \alpha(x, y))\,\mathrm{d}y.$$

(a) Show that X satisfies detailed balance with respect to π, i.e. that $\pi(x)p(x, y) = \pi(y)p(y, x)$ for all $x, y \in \mathbb{R}$.

Suppose that $\kappa(x, y) = f(y)$ for some continuous function $f : \mathbb{R} \to (0, \infty)$ (i.e. we are using an *independence sampler*), and that there exists a constant β with $f(y)/\pi(y) \geq \beta$ for all $y \in \mathbb{R}$.

(b) Explain why X is π-irreducible (you do not need to write down a formal proof).

(c) Show that the whole state space is a small set, and deduce that X is uniformly ergodic.

Now instead suppose that

$$\kappa(x, y) = \begin{cases} 1 & \text{if } |y - x| < 1/2 \\ 0 & \text{otherwise,} \end{cases}$$

i.e. we propose to move uniformly on the unit ball around our current position.

(d) Show that any interval $C = [a, b]$ with $0 < b - a < 1$ is a small set.
HINT: Note that

$$\delta = \frac{1}{\sup_{z \in [a-1/2, b+1/2]} \pi(z)} > 0$$

and try

$$\nu(A) = \frac{\pi(A \cap (b - 1/2, a + 1/2))}{\pi((b - 1/2, a + 1/2))}.$$