APTS Applied Stochastic Processes, Durham, April 2021
Exercise Sheet for Assessment

The work here is "light-touch assessment", intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level *of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

1 Markov chains, reversibility and Poisson processes

Suppose that visitors to a museum arrive as a Poisson process of rate λ. Each visitor spends an $\text{Exp}(\mu)$ time looking around, independently of the other visitors, and then leaves. Let X_t be the number of customers inside the museum at time $t \geq 0$. We assume that $\lambda, \mu > 0$.

(a) Convince yourself that $(X_t)_{t\geq 0}$ is a continuous-time Markov chain, and write down its transition rates.

(b) Find the stationary distribution for $(X_t)_{t\geq 0}$. Do you need any constraints on the parameters λ and μ?

(c) In the COVID-19 pandemic, the museum introduces a policy that at most N people can be inside at any one time. Anyone not allowed in joins a queue outside. The person at the front of the queue enters as soon as someone leaves the museum; otherwise, no-one leaves the queue. Modelling the total number of people (number in the museum plus number in the queue) as a Markov chain $(Y_t)_{t\geq 0}$, what constraints on λ and μ are required to ensure that $(Y_t)_{t\geq 0}$ has a stationary distribution? (You do not need to calculate the stationary distribution in full.)

2 Martingales and optional stopping

(a) Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with a common Poisson distribution having mean $\lambda > 0$. Show that

$$X_n = \exp\big(\alpha(Y_1 + \ldots + Y_n) - n\lambda(\mathrm{e}^{\alpha} - 1)\big)$$

defines a martingale $X_0 = 1, X_1, X_2, \ldots$.

(b) Explain why it is a consequence from martingale theory that X_n converges almost surely as $n \to \infty$. Verify this directly by applying the strong law of large numbers to $\log X_n$, and hence identify the limit.

3 Small sets and ergodicity

Define a Markov chain X taking values in $[1, \infty)$ as follows: for $n \geq 0$, let

$$X_{n+1} = \sqrt{X_n} + E_{n+1},$$

where $E_1, E_2, E_3 \ldots$ are independent exponential random variables of mean 1.

(a) Argue that X is ϕ-irreducible, where ϕ denotes Lebesgue (length) measure on $[1, \infty)$.

(b) Show that any set of the form $C = [1, c]$ is small of lag 1, for $c > 1$.

(c) Show that X is geometrically ergodic. Explain why X is not uniformly ergodic.