APTS ASP Exercises 2019

Markov chains and reversibility

1. Show that a discrete-time Markov chain run backwards in time (from some time n and state i) is again a Markov chain (until time n).

2. Suppose that $p_{x,y}$ are transition probabilities for a discrete state-space Markov chain satisfying detailed balance. Show that if the system of probabilities given by π_x satisfy the detailed balance equations then they must also satisfy the equilibrium equations.

3. Show that unconstrained simple symmetric random walk has period 2. Show that simple symmetric random walk subject to "prohibition" boundary conditions must be aperiodic.

4. Solve the equilibrium equations $\pi P = \pi$ for simple symmetric random walk on $\{0, 1, \ldots, k\}$ subject to "prohibition" boundary conditions.

5. Suppose that $X_0, X_1, \ldots,$ is a simple symmetric random walk with "prohibition" boundary conditions as above.

   - Use the definition of conditional probability to compute
   $$\overline{p}_{y,x} = \frac{\mathbb{P}\left[X_{n-1} = x\,,\, X_n = y\right]}{\mathbb{P}\left[X_n = y\right]}\,,$$
   - then show that
   $$\frac{\mathbb{P}\left[X_{n-1} = x\,,\, X_n = y\right]}{\mathbb{P}\left[X_n = y\right]} = \frac{\mathbb{P}\left[X_{n-1} = x\right] p_{x,y}}{\mathbb{P}\left[X_n = y\right]}\,,$$
   - now substitute, using $\mathbb{P}\left[X_n = i\right] = \frac{1}{k+1}$ for all i so $\overline{p}_{y,x} = p_{x,y}$.
   - Use the symmetry of the kernel ($p_{x,y} = p_{y,x}$) to show that the backwards kernel $\overline{p}_{y,x}$ is the same as the forwards kernel $\overline{p}_{y,x} = p_{y,x}$.

6. Show that if $X_0, X_1, \ldots,$ is a simple *asymmetric* random walk with "prohibition" boundary conditions, running in equilibrium, then it also has the same statistical behaviour as its reversed chain (i.e. solve the detailed balance equations!).

7. Show that detailed balance doesn't work for the 3-state chain with transition probabilities $\frac{1}{3}$ for $0 \to 1$, $1 \to 2$, $2 \to 0$ and $\frac{2}{3}$ for $2 \to 1$, $1 \to 0$, $0 \to 2$.

8. Use Burke's theorem for a feed-forward $\cdot/M/1$ queueing network (no loops) to show that in equilibrium each queue viewed in isolation is $M/M/1$. This uses the fact that independent thinnings and superpositions of Poisson processes are still Poisson ....

9. Work through the Random Chess example to compute the mean return time to a corner of the chessboard.

10. Verify for the Ising model that
$$\mathbb{P}\left[\mathbf{S} = \mathbf{s}^{(i)} \middle| \mathbf{S} \in \{\mathbf{s}, \mathbf{s}^{(i)}\}\right] = \frac{\exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}{\exp\left(J \sum_{j:j\sim i} s_i s_j\right) + \exp\left(-J \sum_{j:j\sim i} s_i s_j\right)}\,.$$
    Determine how this changes in the presence of an external field. Confirm that detailed balance holds for the heat-bath Markov chain.

11. Write down the transition probabilities for the Metropolis-Hastings sampler. Verify that it has the desired probability distribution as an equilibrium distribution.

Renewal processes and stationarity

1. Suppose that X is a simple symmetric random walk on $\mathbb{Z}$, started from 0. Show that

$$T = \inf\{n \geq 0 : X_n \in \{-10, 10\}\}$$

is a stopping time (i.e. show that the event $\{T \leq n\}$ is determined by $X_0, X_1, \ldots, X_n$). What is the value of $\mathbb{P}[T < \infty]$? What is the distribution of X_T?

2. For a Markov chain $(X_n)_{n\geq 0}$ on a state-space S, fix $i \in S$ and let $H_0^{(i)} = \inf\{n \geq 0 : X_n = i\}$. For $m \geq 0$, let

$$H_{m+1}^{(i)} = \inf\{n > H_m^{(i)} : X_n = i\}.$$

Show that $H_0^{(i)}, H_1^{(i)}, \ldots$ is a sequence of stopping times.

3. Check that it follows from the Strong Markov property that $(H_{m+1}^{(i)} - H_m^{(i)}, m \geq 0)$ is a sequence of i.i.d. random variables, independent of $H_0^{(i)}$.

4. Suppose that $(N(n))_{n\geq 0}$ is a delayed renewal process with inter-arrival times $Z_0, Z_1, \ldots$ where Z_0 is a non-negative random variable, independent of $Z_1, Z_2, \ldots$ which are i.i.d. strictly positive random variables with common mean μ. Use the Strong Law of Large Numbers for $T_k = \sum_{i=0}^k Z_i$ to show that

$$\frac{N(n)}{n} \to \frac{1}{\mu} \qquad \text{a.s. as } n \to \infty.$$

Hint: note that $T_{N(n)} \leq n < T_{N(n)+1}$ *so that* $N(n)/n$ *can be sandwiched between* $N(n)/T_{N(n)+1}$ *and* $N(n)/T_{N(n)}$. *Use this and the fact that* $N(n) \to \infty$ *as* $n \to \infty$.

5. Let $(Y(n))_{n\geq 0}$ be the auxiliary Markov chain associated to a delayed renewal process $(N(n))_{n\geq 0}$ i.e. $Y(n) = T_{N(n-1)} - n$. Check that you agree with the transition probabilities given in the lecture notes.

6. Let

$$\nu_i = \frac{1}{\mu}\mathbb{P}[Z_1 \geq i+1], \quad i \geq 0.$$

Check that $\nu = (\nu_i)_{i\geq 0}$ defines a probability mass function.

7. Suppose that Z^* has the *size-biased distribution* associated with the distribution of Z_1, defined by

$$\mathbb{P}[Z^* = i] = \frac{i\,\mathbb{P}[Z_1 = i]}{\mu}, \quad i \geq 1.$$

   (a) Verify that this is a probability mass function.

   (b) Given $Z^* = k$, let $L \sim \mathrm{U}\{0, 1, \ldots, k-1\}$. Show that, unconditionally, $L \sim \nu$.
   Note that you can generate L *starting from* Z^* *by letting* $U \sim U[0,1]$ *and then setting* $L = \lfloor UZ^* \rfloor$.

   (c) What is the size-biased distribution associated with $\mathrm{Po}(\lambda)$?

8. Show that ν is stationary for Y.
Hint: Y *is clearly not reversible, so there's no point trying detailed balance!*

9. Check that if $\mathbb{P}[Z_1 = k] = (1-p)^{k-1}p$, for $k \geq 1$, the stationary distribution ν for the time until the next renewal is $\nu_i = (1-p)^i p$, for $i \geq 0$. (In other words, if we flip a biased coin with probability p of heads at times $n = 0, 1, 2, \ldots$ and let $N(n) = \#\{0 \leq k \leq n : \text{we see a head at time } k\}$ then $(N(n), n \geq 0)$ is a stationary delayed renewal process.)

Martingales

1. Let X be a martingale. Use the tower property for conditional expectation to deduce that

$$\mathbb{E}\left[X_{n+k}|\mathcal{F}_n\right] = X_n\,, \quad k = 0, 1, 2, \ldots\,.$$

2. Recall Thackeray's martingale: let $Y_1, Y_2, \ldots$ be a sequence of independent random variables, with $\mathbb{P}[Y_1 = 1] = \mathbb{P}[Y_1 = -1] = 1/2$. Define the Markov chain M by

$$M_0 = 0; \qquad M_n = \begin{cases} 1 - 2^n & \text{if } Y_1 = Y_2 = \cdots = Y_n = -1, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathbb{E}[M_n]$ from first principles.

   (b) What should be the value of $\mathbb{E}\left[\widetilde{M}_n\right]$ if $\widetilde{M}$ is computed as for M but stopping play if M hits level $1 - 2^N$?

3. Consider a branching process Y, where $Y_0 = 1$ and Y_{n+1} is the sum $Z_{n+1,1} + \ldots + Z_{n+1,Y_n}$ of Y_n independent copies of a non-negative integer-valued family-size r.v. Z.

   (a) Suppose $\mathbb{E}[Z] = \mu < \infty$. Show that $X_n = Y_n/\mu^n$ is a martingale.

   (b) Show that Y is itself a supermartingale if $\mu < 1$ and a submartingale if $\mu > 1$.

   (c) Suppose $\mathbb{E}\left[s^Z\right] = G(s)$. Let η be the smallest non-negative root of the equation $G(s) = s$. Show that η^{Y_n} defines a martingale.

   (d) Let $H_n = Y_0 + \ldots + Y_n$ be the total of all populations up to time n. Show that $s^{H_n}/(G(s)^{H_{n-1}})$ is a martingale.

   (e) How should these three expressions be altered if $Y_0 = k \geq 1$?

4. Consider asymmetric simple random walk, stopped when it first returns to 0. Show that this is a supermartingale if jumps have non-positive expectation, a submartingale if jumps have non-negative expectation (and therefore a martingale if jumps have zero expectation).

5. Consider Thackeray's martingale based on asymmetric random walk. Show that this is a supermartingale or submartingale depending on whether jumps have negative or positive expectation.

6. Show, using the conditional form of Jensen's inequality, that if X is a martingale then $|X|$ is a submartingale.

7. A shuffled pack of cards contains b black and r red cards. The pack is placed face down, and cards are turned over one at a time. Let B_n denote the number of black cards left *just before* the n^{th} card is turned over. Let

$$Y_n = \frac{B_n}{r + b - (n-1)}\,.$$

   (So Y_n equals the proportion of black cards left just before the n^{th} card is revealed.) Show that Y is a martingale.

8. Suppose N_1, N_2, ... are independent identically distributed normal random variables of mean 0 and variance σ^2, and put $S_n = N_1 + \ldots + N_n$.

   (a) Show that S is a martingale.

   (b) Show that $Y_n = \exp\left(S_n - \frac{n}{2}\sigma^2\right)$ is a martingale.

   (c) How should these expressions be altered if $\mathbb{E}[N_i] = \mu \neq 0$?

9. Let X be a discrete-time Markov chain on a countable state-space S with transition probabilities $p_{x,y}$. Let $f : S \to \mathbb{R}$ be a bounded function. Let $\mathcal{F}_n$ contain all the information about $X_0, X_1, \ldots, X_n$. Show that

$$M_n = f(X_n) - f(X_0) - \sum_{i=0}^{n-1} \sum_{y \in S} (f(y) - f(X_i)) p_{X_i,y}$$

defines a martingale. (Hint: first note that $\mathbb{E}\left[f(X_{i+1}) - f(X_i)|X_i\right] = \sum_{y\in S}(f(y) - f(X_i))p_{X_i,y}$. Using this and the Markov property of X, check that $\mathbb{E}\left[M_{n+1} - M_n|\mathcal{F}_n\right] = 0$.)

10. Let Y be a discrete-time birth-death process absorbed at zero:

$$p_{k,k+1} = \frac{\lambda}{\lambda+\mu}, \quad p_{k,k-1} = \frac{\mu}{\lambda+\mu}, \qquad \text{for } k > 0, \text{ with } 0 < \lambda < \mu.$$

   (a) Show that Y is a supermartingale.

   (b) Let $T = \inf\{n : Y_n = 0\}$ (so $T < \infty$ a.s.), and define

$$X_n = Y_{n\wedge T} + \left(\frac{\mu-\lambda}{\mu+\lambda}\right)(n \wedge T).$$

   Show that X is a non-negative supermartingale, converging to

$$Z = \left(\frac{\mu-\lambda}{\mu+\lambda}\right)T.$$

   (c) Deduce that

$$\mathbb{E}\left[T|Y_0 = y\right] \le \left(\frac{\mu+\lambda}{\mu-\lambda}\right)y.$$

11. Let $L(\theta; X_1, X_2, \ldots, X_n)$ be the likelihood of parameter θ given a sample of independent and identically distributed random variables, $X_1, X_2, \ldots, X_n$.

   (a) Check that if the "true" value of θ is θ_0 then the likelihood ratio

$$M_n = \frac{L(\theta_1; X_1, X_2, \ldots, X_n)}{L(\theta_0; X_1, X_2, \ldots, X_n)}$$

   defines a martingale with $\mathbb{E}\left[M_n\right] = 1$ for all $n \ge 1$.

   (b) Using the strong law of large numbers and Jensen's inequality, show that

$$\frac{1}{n}\log M_n \to -c \text{ as } n \to \infty.$$

12. Let X be a simple symmetric random walk absorbed at boundaries $a < b$.

   (a) Show that

$$f(x) = \frac{x-a}{b-a} \qquad x \in [a, b]$$

   is a bounded harmonic function.

   (b) Use the martingale convergence theorem and optional stopping theorem to show that

$$f(x) = \mathbb{P}\left[X \text{ hits } b \text{ before } a|X_0 = x\right].$$

Recurrence and rates of convergence

1. Recall that the total variation distance between two probability distributions μ and ν on $\mathcal{X}$ is given by

$$\text{dist}_{\text{TV}}(\mu, \nu) = \sup_{A\subseteq\mathcal{X}}\{\mu(A) - \nu(A)\}.$$

   Show that this is equivalent to the distance (note the absolute value signs!)

$$\sup_{A\subseteq\mathcal{X}}|\mu(A) - \nu(A)|.$$

2. Show that if $\mathcal{X}$ is discrete, then

$$\mathrm{dist}_{\mathrm{TV}}(\mu, \nu) = \tfrac{1}{2} \sum_{y \in \mathrm{X}} |\mu(y) - \nu(y)| \,.$$

(Here we *do* need to use the absolute value on the RHS!)
Hint: consider $A = \{y : \mu(y) > \nu(y)\}$.

3. Suppose now that μ and ν are density functions on $\mathbb{R}$. Show that

$$\mathrm{dist}_{\mathrm{TV}}(\mu, \nu) = 1 - \int_{-\infty}^{\infty} \min\{\mu(y), \nu(y)\} dy \,.$$

Hint: remember that $|\mu - \nu| = \mu + \nu - 2\min\{\mu, \nu\}$.

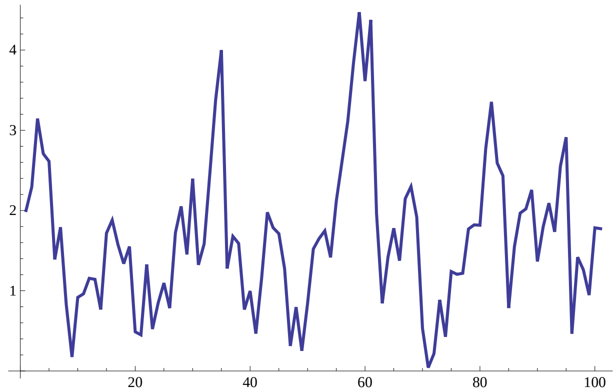
4. Let X be a random walk on $\mathbb{R}$, with increments given by the standard normal distribution. Recall that any bounded set is small of lag 1. Does there exist $k \geq 1$ such that *the whole state space* is small of lag k?

5. Consider a Markov chain X with continuous transition density kernel. Show that it possesses *many* small sets of lag 1.

6. Consider a Vervaat perpetuity X, where

$$X_0 = 0; \qquad X_{n+1} = U_{n+1}(X_n + 1) \,,$$

and where $U_1, U_2, \ldots$ are independent Uniform$(0, 1)$ (simulated below).



Find a small set for this chain.

7. Recall the idea of regenerating when our chain hits a small set: suppose that C is a small set (with lag 1) for a ϕ-recurrent chain X, i.e. for $x \in C$,

$$\mathbb{P}[X_1 \in A | X_0 = x] \geq \alpha\nu(A).$$

Suppose that $X_n \in C$. Then with probability α let $X_{n+1} \sim \nu$, and otherwise let it have transition distribution $\frac{p(x,\cdot) - \alpha\nu(\cdot)}{1-\alpha}$.

   (a) Check that the latter expression really gives a probability distribution.

   (b) Check that X_{n+1} constructed in this manner obeys the correct transition distribution from X_n.

8. Define a reflected random walk as follows: $X_{n+1} = \max\{X_n + Z_{n+1}, 0\}$, for $Z_1, Z_2, \ldots$ i.i.d. with continuous density $f(z)$,

$$\mathbb{E}[Z_1] < 0 \quad \text{and} \quad \mathbb{P}[Z_1 > 0] > 0 \,.$$

Show that the Foster-Lyapunov criterion for positive recurrence holds, using $\Lambda(x) = x$.