

## Computer Practical 2

# Markov Chains and Monte Carlo

*The first problem sheet contained several problems and if you want to continue working on those that is fine; this sheet just contains a couple of simple questions to give you a chance to try some Markov chain-based problems.*

1. *A warm-up which also appeared in the preliminary material; if you've never implemented something like this before then this might be a useful preliminary step.*

In a simplified model of the game of Monopoly, we consider the motion of the piece around a loop of 40 spaces. We can model this as a Markov chain with a state space consisting of the integers  $0, \dots, 39$  in which the transition kernel adds the result of two six-sided dice to the current state modulo 40 to obtain the new state.

- (a) Implement a piece of R code which simulates this Markov chain.
- (b) Run the code for a large number of iterations, say 100,000, and plot a histogram of the states visited.
- (c) Based on the output of the chain, would you conjecture that there is an invariant distribution for this Markov chain? If so, what?
- (d) Write the transition kernel down mathematically.
- (e) Check whether the Markov kernel you have written down is invariant with respect to any distribution conjectured in part (c).

2. *An actual Gibbs Sampler.*

Recall the Poisson changepoint model discussed in lectures, and on p28 of the supporting notes, and think about the following closely related model: Observations  $y_1, \dots, y_n$  comprise a sequence of  $M$  iid  $\mathbf{N}(\mu_1, 1)$  random variables followed by a second sequence of  $n - M$  iid  $\mathbf{N}(\mu_2, 1)$  random variables.  $M$ ,  $\mu_1$  and  $\mu_2$  are unknown. The prior distribution over  $M$  is a discrete uniform distribution on  $\{1, \dots, n - 1\}$  (there is at least one observation of each component). The prior distribution over  $\mu_i$  ( $i = 1, 2$ ) is  $\mathbf{N}(0, 10^2)$ . The three parameters are treated as being a priori independent.

- (a) Write down the joint density of  $y_1, \dots, y_n, \mu_1, \mu_2$  and  $M$ , and obtain the posterior distribution of  $\mu_1, \mu_2$  and  $M$ , up to proportionality, in as simple a form as you can.
- (b) Find the “full conditional” distributions of  $\mu_1$ ,  $\mu_2$  and  $M$ . (i.e. the conditional distributions of each of these variables given all other variables).
- (c) Implement a Gibbs sampler making use of these full conditional distributions in order to target the posterior distribution identified in part (b).
- (d) Simulate some data from the model for various parameter values and test your Gibbs sampler.
- (e) How might you extend this algorithm if instead of a changepoint model you had a mixture model in which every observation is drawn from a mixture, i.e.:

$$Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} p\mathbf{N}(\cdot; \mu_1, 1) + (1 - p)\mathbf{N}(\cdot; \mu_2, 1).$$

(The likelihood is now  $\prod_{i=1}^n [p\mathbf{N}(y_i; \mu_1, 1) + (1 - p)\mathbf{N}(y_i; \mu_2, 1)]$ , with  $p$ ,  $\mu_1$ , and  $\mu_2$  unknown (and  $M$  is no longer a parameter of the model).)

Consider the following things:

- (i) The prior distribution over  $p$ .
- (ii) Any other variables you may need to introduce.
- (iii) The resulting algorithm.

If you have time, implement the resulting algorithm and apply it to some simulated data.