

# APTS Applied Stochastic Processes, Durham, April 2025

## Exercise Sheet for Assessment

*The work here is “light-touch assessment”, intended to take students up to half a week to complete. Students should talk to their supervisors to find out whether or not their department requires this work as part of any formal accreditation process (APTS itself has no resources to assess or certify students). It is anticipated that departments will decide the appropriate level of assessment locally, and may choose to drop some (or indeed all) of the parts, accordingly.*

Students are recommended to read through the relevant portion of the lecture notes before attempting each question.

### 1 Markov chains and reversibility

Emails arrive in a researcher’s inbox at rate  $\alpha > 0$ . The researcher deals with tasks at rate  $\beta X$ , if there are currently  $X$  emails in the inbox. An email which is dealt with is then immediately removed from the inbox. Interest is focussed on the proportion of time for which the researcher’s inbox is empty, as this presumably corresponds to periods of undistracted productivity.

- (a) Model this situation using a reversible continuous-time Markov chain and produce a formula for the equilibrium probability of the researcher’s inbox being empty.
- (b) The researcher’s manager considers that the researcher is not obtaining sufficient undistracted research time. Options are:
  - (i) send researcher on course “Dealing with your inbox”, which will double the rate at which the researcher deals with emails;
  - (ii) instruct the researcher to follow the protocol, delete on arrival all emails without exception which arrive when there are at least  $K$  emails currently in the inbox.

Under which circumstances might option (ii) be better than option (i)? (Illustrate your answer in the case in which the arrival rate of emails is seven times the rate with which the researcher deals with an individual email.)

- (c) Write down a general formula for the equilibrium probability with which an arriving email is immediately deleted on arrival, if option (ii) is adopted.

### 2 Martingales and Optional stopping

A shuffled pack of cards contains  $b \geq 1$  black and  $r \geq 1$  red cards. The pack is placed face down on a table, and cards are turned over one at a time. Let  $B_n$  denote the number of black cards left *just before* the  $n^{\text{th}}$  card is turned over. (So, for example,  $B_1 = b$ .)

- (a) Show that

$$M_n = \frac{B_n}{r + b - (n - 1)},$$

the proportion of black cards left just before the  $n^{\text{th}}$  card is revealed, defines a martingale.

- (b) Let  $T$  be the time at which the first black card is turned over. Compute the probability mass function of this random variable.

- (c) Use part (b) and the Optional stopping theorem applied to  $M_{T+1}$  to show that, for  $b \geq 2$  and  $r \geq 1$ ,

$$\sum_{k=0}^r \frac{\binom{r}{k}}{\binom{r+b-2}{k}} = \frac{r+b-1}{b-1}.$$

### 3 Small sets and Foster–Lyapunov criteria

Define a Markov chain  $X$  taking values in  $[1, \infty)$  as follows: for  $n \geq 0$ , let

$$X_{n+1} = \sqrt{X_n} + E_{n+1},$$

where  $E_1, E_2, E_3 \dots$  are independent exponential random variables of mean 1.

- (a) Argue that  $X$  is  $\phi$ -irreducible, where  $\phi$  denotes Lebesgue (length) measure on  $[1, \infty)$ .
- (b) Show that any set of the form  $C = [1, c]$  with  $c > 1$  is a small set.
- (c) Show that  $X$  is geometrically ergodic. Explain why  $X$  is not uniformly ergodic.