AEA 2002 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. It is important in a question like this to get off to the right start — trigonometric identities abound, and it is not immediately transparent which one to use in order to simplify the given equation and then solve it. One way of reasoning as to the right approach is to recognise that, in solving an equation $C(x) = 0$, it is very beneficial if one can factor it out to $A(x)B(x) = 0$ for some functions A and B. It then follows that x solves the original equation precisely when x solves $A(x) = 0$ or x solves $B(x) = 0$. The simplification is achieved if the functions A and B are more amenable to further analysis than C. Note here that solutions to both $A(x) = 0$ as well as $B(x) = 0$ are solutions to the original equation, rather than just one or the other!

   To obtain the C in our case, we first carry over the terms in the equation $\sin(5x)-\cos(5x) = \cos(x) - \sin(x)$ to one side:

   $$\sin(5x) + \sin(x) - (\cos(5x) + \cos(x)) = 0.$$

   Thus the function C includes sums of sines and cosines, and going from sums to products is achieved by using the sum-to-product rule formulae for the sine and cosine:

   $$\cos(\theta) + \cos(\phi) = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

   and

   $$\sin(\theta) + \sin(\phi) = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right).$$

   In particular, using these sum-to-product rules, the equation becomes

   $$2\sin(3x)\cos(2x) - 2\cos(3x)\cos(2x) = 0.$$

   Cancelling 2 and factorising then gives:

   $$\cos(2x)(\sin(3x) - \cos(3x)) = 0.$$

   As per the discussion in the first paragraph, we obtain solutions to the equation when either $\cos(2x) = 0$ or $\sin(3x) = \cos(3x)$. For the first of these options, $\cos(2x) = 0$ is equivalent to $2x = \pi/2 + k\pi$ with k an integer. For the second, we start by noting that if $\cos(3x) = 0$, then $\sin(3x)$ is either 1 or -1, i.e. there is no solution x to $\sin(3x) = \cos(3x)$ with $\cos(3x) = 0$. We can therefore assume that $\cos(3x) \neq 0$, and divide by it to obtain that the condition is equivalent to $\tan(3x) = 1$, which means $3x = \pi/4 + m\pi$ with m an integer. Finally, since $0 \leq x \leq \pi$, it follows that $\pi/4$, $3\pi/4$, $\pi/12$ and $5\pi/12$ are the solutions to the equation given (where the $9\pi/12$ already appears as $3\pi/4$).

2. Recall the binomial expansion:

   $$(1+a)^p = \sum_{k=0}^{\infty} \binom{p}{k} a^k,$$

   where

   $$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!},$$

and a is a real number with $|a| < 1$. Hence, taking $a = -4x$, we have that $(1-4x)^p$ is equal to

$$1+p(-4x)+\frac{p(p-1)}{2!}(-4x)^2+\frac{p(p-1)(p-2)}{3!}(-4x)^3+\frac{p(p-1)(p-2)(p-3)}{4!}(-4x)^4+\cdots$$

For the coefficient of x^2 to be equal to that of x^4, we therefore require that

$$\frac{p(p-1)4^2}{2!} = \frac{p(p-1)(p-2)(p-3)4^4}{4!}. \tag{1}$$

For the coefficient of x^3 to be positive, we need

$$\frac{p(p-1)(p-2)(-4)^3}{3!} > 0. \tag{2}$$

Due to (2), it can not be the case that p is equal to 0 or 1 (otherwise the left-hand side of (2) would be 0, which contradicts that it is strictly greater than 0). Hence, we can divide (1) by $4^2p(p-1)$ to obtain

$$\frac{1}{2} = \frac{16(p-2)(p-3)}{24}.$$

Rearranging this gives that

$$0 = 4p^2 - 20p + 21,$$

which can be factorised as follows

$$0 = (2p-3)(2p-7).$$

Hence $p = 3/2$ or $p = 7/2$. However if $p = 7/2$, then the left-hand side of (2) is negative. Thus it must be the case that $p = 3/2$.

3. The question implies that the point $(14, 1)$ lies on the curve C. It will be helpful to know which value of t it corresponds to. To deduce this, we need to solve

$$14 = 15t - t^3 \quad \text{and} \quad 1 = 3 - 2t^2,$$

simultaneously. The second relation is easily rearranged to $t^2 = 1$, so that t is either -1 or 1. The only choice from these that also satisfies the first relation is $t = 1$.

Now, to find the normal at to the curve at $(14, 1)$, we will start by computing its derivative there. For a curve given in a parametric form, as is the case here, we can do this by using the following rule:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

In particular, since

$$\frac{dx}{dt} = 15 - 3t^2 \quad \text{and} \quad \frac{dy}{dt} = -4t,$$

it follows that

$$\left.\frac{dy}{dx}\right|_{(x,y)=(14,1)} = \left.\frac{-4t}{15-3t^2}\right|_{t=1} = -\frac{1}{3}.$$

Given that the slope of the tangent at $(14, 1)$ is $k = -1/3$, the slope of the normal must be $k' := -1/k = 3$, and its equation is therefore given by $y - 1 = k'(x - 14)$, i.e.

$$y = 3x - 41. \tag{3}$$

We now need to find where this line cuts C. Substituting the original parametric equations ($x = 15t - t^3$ and $y = 3 - 2t^2$) into (3) yields:

$$3 - 2t^2 = 3(15t - t^3) - 41,$$

which is a cubic in t, alternatively expressed as

$$3t^3 - 2t^2 - 45t + 44 = 0. \tag{4}$$

With a cubic, it always makes sense to check if we can find one solution by a trial-and-guess method, since explicit expressions for cubics, whilst known, are quite complicated. In fact, we already know that 1 is a root, since obviously the normal to C at $(14, 1)$ cuts C there, and this corresponds to setting $t = 1$; a quick calculation confirms that this choice of t does indeed solve (4). Thus we can factor $t - 1$ out in the left-hand side by polynomial long division as follows:

$$\begin{aligned} 3t^3 - 2t^2 - 45t + 44 &= (t-1)3t^2 + t^2 - 45t + 44 \\ &= (t-1)3t^2 + (t-1)t - 44t + 44 \\ &= (t-1)\left(3t^2 + t - 44\right). \end{aligned}$$

Thus, apart from $t = 1$, the intersections of the normal and C are given by the equation

$$3t^2 + t - 44 = 0.$$

This quadratic in t is most easily solved directly by employing the formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 3$, $b = 1$ and $c = -44$. Thus, the two solutions are

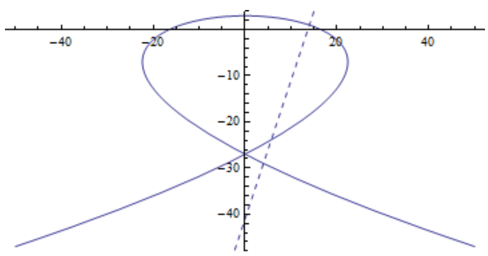
$$t = \frac{-1 + \sqrt{1 + 12 \cdot 44}}{6} = \frac{-1 + 23}{6} = \frac{11}{3}$$

and

$$t = \frac{-1 - 23}{6} = -4.$$

Hence the sought after values of t constitute the set $\{-4, 11/3\}$ (we do not include 1, because we are looking for the points where the normal cuts C *again*).

Although we did not need it to solve the question, it is perhaps enlightening to see what the curve C corresponds to. This is shown in the following figure, in which the dotted line is the relevant normal.



4. The curve $y(x)$ is given in implicit form. To find its stationary points, we need to find the values of x for which $y'(x) = 0$. To do this, we will try differentiating the equation we are given. In particular, if we differentiate both sides of the identity

$$x^3 + y(x)^3 - 3xy(x) = 48 \tag{5}$$

with respect to x, using the product and chain rule, we obtain

$$3x^2 + 3y(x)^2 y'(x) - 3xy'(x) - 3y(x) = 0. \tag{6}$$

Since we are looking for values of x where $y'(x) = 0$, we set $y'(x) = 0$ in the above equation to give that $3x^2 = 3y$, i.e. $y = x^2$. Now, plug this back into the original implicit equation for the curve (5) to get

$$x^3 + x^6 - 3x^3 = 48.$$

This might look complicated, but we notice that it is actually just a quadratic equation in $u = x^3$. In particular, $u^2 - 2u - 48 = 0$. This is easily factorised to

$$(u - 8)(u + 6) = 0.$$

Thus $x^3 = u = 8$ or $x^3 = u = -6$. On the real numbers, the map $x \mapsto x^3$ is one-to-one, with inverse given by the cubic root. Hence $x = 2$ or $x = -6^{1/3}$ are the only possible values for x at which y is stationary. As noted above, the corresponding values of y satisfy $y = x^2$, and so are equal to 4 and $6^{2/3}$, respectively. Finally, we need to check that $y'(x) = 0$ does indeed hold at these points. To do this, we first substitute the values $(x, y) = (2, 4)$ back into (6) to obtain

$$12 + 48y'(x) - 6y'(x) - 12 = 0.$$

Clearly this implies $y'(x) = 0$, as desired. We may proceed similarly for the point $(x, y) = (-6^{1/3}, 6^{2/3})$. (Of course, one should also check that the points $(2, 4)$ and $(-6^{1/3}, 6^{2/3})$ in fact lie on the curve, by checking the validity of (5) at these points)

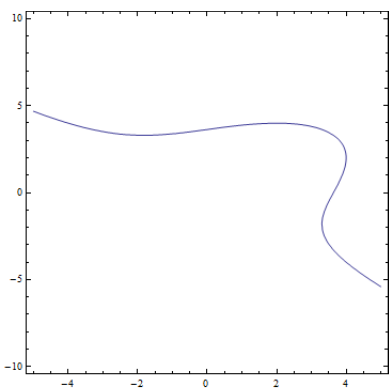
To establish the nature of the stationary points, we will investigate the second derivative. So, differentiate the identity (6) again to get:

$$6x + 6y(x)y'(x)^2 + 3y(x)^2 y''(x) - 3y'(x) - 3xy''(x) - 3y'(x) = 0. \tag{7}$$

Now use the fact that $y'(x) = 0$ at a stationary point, to get

$$y''(x) = \frac{2x}{x - y(x)^2}.$$

The point $(2, 4)$ has $y''(x) = -2 < 0$, hence this is a local maximum. The point $(-6^{1/3}, 6^{2/3})$ has $y''(x) = 2 \cdot 6^{1/3}/(6^{1/3} + 6^{4/3}) > 0$, hence this is a local minimum. This is nicely seen in the following figure.

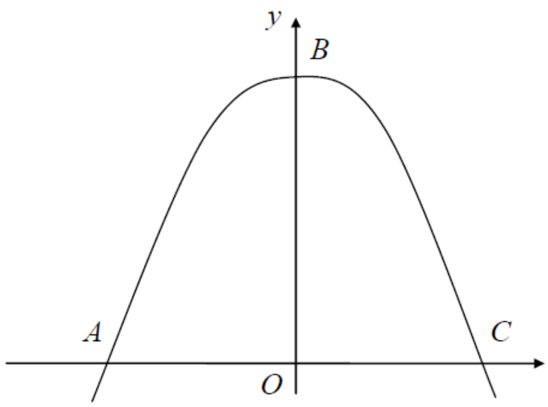


(Note that the figure also shows that the equation does not in fact define a function $x \mapsto y(x)$ uniquely – for some values of x, we can choose from multiple values for y.)

5. (a) The coordinates of the points A and C are obtained by setting $y = 0$ in the equation $y = \sin(\cos(x))$. Thus $\sin(\cos(x)) = 0$ which requires $\cos(x) = k\pi$ for some integer k. Since $\cos(x)$ takes values in between -1 and 1 for all x and $\pi > 1$, it follows that $k = 0$ is the only possibility. It follows that $x = \frac{\pi}{2} + m\pi$ for some integer m. From the figure given in the question (see below), we see that A and C are the first points of intersection with the x-axis, in the negative and positive x directions, respectively. Therefore,

$$A = \left(-\frac{\pi}{2}, 0\right) \quad \text{and} \quad C = \left(\frac{\pi}{2}, 0\right).$$

The coordinates of B are obtained by setting $x = 0$, thus $y = \sin(\cos(0)) = \sin(1)$, hence $C = (0, \sin(1))$.



(b) To check that B is a stationary point, we will show that $y'(x) = 0$ there. Differentiating using the chain rule gives

$$y'(x) = \cos(\cos(x))(-\sin(x)).$$

Hence $y'(0) = 0$, since $\sin(0) = 0$. It follows that B is indeed a stationary point.

(c) For all $\theta \geq 0$, $\sin\theta \leq \theta$, with equality only if $\theta = 0$. Therefore for all $x \in [0, \pi/2]$,

$$\sin(\cos(x)) \leq \cos(x),$$

with equality only if $x = \pi/2$ (since this is the only x, for which $\cos(x) = 0$ on $x \in [0, \pi/2]$).

For the second relation observe that by convexity the line BC lies below the curve. (There is nothing profound here in trying out this fact, except for the hint that one should use convexity in some fashion. The fact that one has $\sin(1)$ and $\pi/2$ entering the relation that we are trying to show, however, gives some indication that the points B and C might be a suitable choice! We would arrive at the same result by using instead the points B and A.) Writing down the equation for the line which passes through B and C is easy,

$$y - \sin(1) = \frac{\sin(1) - 0}{0 - \pi/2}(x - 0).$$

Therefore, by the observation made above about this line lying beneath the curve,

$$\sin(1) - \frac{\sin(1)}{\pi/2}x \leq \sin(\cos(x)),$$

and rearranging gives

$$\left(1 - \frac{2}{\pi}x\right)\sin(1) \leq \sin(\cos(x)),$$

as required. One has equality only when the line BC intersects the curve, hence at $x = 0$ and $x = \pi/2$.

(d) By comparing the areas under the relevant curves, the two relations established in part (c) yield immediately that:

$$I_1 := \int_0^{\pi/2} \left(1 - \frac{2}{\pi}x\right)\sin(1)dx < \int_0^{\pi/2} \sin(\cos(x))dx < \int_0^{\pi/2} \cos(x)dx =: I_2.$$

Moreover, the integrations in I_1 and I_2 are straightforward. In particular,

$$I_1 = \frac{\pi}{4}\sin(1)$$

and

$$I_2 = \sin(\pi/2) - \sin(0) = 1,$$

and hence the required result. (Notice that I_1 could also have been computed as the area of a triangle.)

6. (a) In the question, we are given a lot of information, and so we will start by summarising what this means for the values of m_1, n_1, m_2, n_2:

- $m_2 > m_1$ and m_1, n_1, m_2, n_2 are all positive integers.
- Symmetry about the line $x = 0$ (and the first condition) means that n_1 and n_2 are both even.
- The points $(\pm 3, 0)$ lie on both curves, or equivalently $m_1 = 3^{n_1}$ and $m_2 = 3^{n_2}$. (In conjunction with the condition $m_2 > m_1$, this implies $n_2 > n_1$.)
- $n_1 + n_2 = 12$.

From the first and second conditions, we know that n_1 and n_2 take values from $2, 4, 6, 8, 10, 12$. The only possible pairs that do this and also satisfy the third and fourth conditions are $(n_1, n_2) = (2, 10)$ and $(n_1, n_2) = (4, 8)$. None of the conditions above rules out either of these choices, and so they are both possible.

(b) The area between C_1 and C_2 is obtained by subtracting the area below C_1 from the area below C_2. Hence, taking into account the symmetry about the line $x = 0$ to get the factor 2 and save up on some of the algebra,

$$\begin{aligned} A &:= 2\int_0^3 (m_2 - x^{n_2})\,dx - 2\int_0^3 (m_1 - x^{n_1})\,dx \\ &= 2\left[m_2 x - \frac{x^{n_2+1}}{n_2+1}\right]_0^3 - 2\left[m_1 x - \frac{x^{n_1+1}}{n_1+1}\right]_0^3 \\ &= 6(m_2 - m_1) + 2\left(\frac{3^{n_1+1}}{n_1+1} - \frac{3^{n_2+1}}{n_2+1}\right). \end{aligned}$$

Now, since $3^{n_1} = m_1$ and $3^{n_2} = m_2$ (by the third condition in part (a)), this reduces to

$$A = 6(m_2 - m_1) + 2\left(\frac{3m_1}{n_1+1} - \frac{3m_2}{n_2+1}\right) = 6\left(m_2\frac{n_2}{n_2+1} - m_1\frac{n_1}{n_1+1}\right).$$

By part (a), we have the choice $(n_1, n_2) = (2, 10)$ or $(n_1, n_2) = (4, 8)$. Clearly as n_1 goes from 2 to 4, $m_1\frac{n_1}{n_1+1}$ increases. Conversely, as n_2 goes from 10 to 8, $m_2\frac{n_2}{n_2+1}$ decreases. Therefore the smaller of the two areas will be achieved by the second choice, and then

$$A = 6 \cdot \left(3^8\frac{8}{9} - 3^4\frac{4}{5}\right) = 2 \cdot 3^5\left(9 \cdot 8 - \frac{4}{5}\right) = 3^5\frac{2 \cdot 356}{5} = \frac{712}{5} \times 3^5 = \frac{173,016}{5}.$$

(c) Equating the gradient means

$$-n_1 x^{n_1 - 1} = -n_2 x^{n_2 - 1}.$$

Excluding the solution $x = 0$, this is equivalent to

$$x^{n_2 - n_1} = \frac{n_1}{n_2},$$

or

$$x = \sqrt[n_2 - n_1]{\frac{n_1}{n_2}}.$$

So, with $(n_1, n_2) = (2, 10)$, one has

$$x = \sqrt[8]{\frac{1}{5}},$$

and with $(n_1, n_2) = (4, 8)$, one has

$$x = \sqrt[4]{\frac{1}{2}}.$$

We now just have to decide which of these values is larger. To check this, we note that taking the 8th power of the first option gives $1/5$, and taking the 8th power of the second option gives $1/4$. Since $1/5$ is smaller than $1/4$, it follows that the largest possible value of x at which the gradients are the same is $\sqrt[4]{1/2}$.

7. (a) The error occurs in line 3, where the form of the argument is

$$pq = 1/2 \Rightarrow (p = 1/2 \text{ or } q = 1/2),$$

which is clearly false in general.

(b) Since we know $1/2$ is a root, we may proceed to factor it out by polynomial long division:

$$\begin{aligned}
x^3 + \frac{3}{4}x - \frac{1}{2} &= \left(x - \frac{1}{2}\right)x^2 + \frac{1}{2}x^2 + \frac{3}{4}x - \frac{1}{2} \\
&= \left(x - \frac{1}{2}\right)x^2 + \left(x - \frac{1}{2}\right)\frac{1}{2}x + \frac{1}{4}x + \frac{3}{4}x - \frac{1}{2} \\
&= \left(x - \frac{1}{2}\right)x^2 + \left(x - \frac{1}{2}\right)\frac{1}{2}x + \left(x - \frac{1}{2}\right) \\
&= \left(x - \frac{1}{2}\right)\left(x^2 + \frac{1}{2}x + 1\right).
\end{aligned}$$

Since the discriminant of $x^2 + x/2 + 1$ is $D = (1/2)^2 - 4 \cdot 1 \cdot 1 < 0$ (using the formula $D = b^2 - 4ac$ for the discriminant of $ax^2 + bx + c$), it follows that $x^2 + x/2 + 1$ has no real roots and hence $1/2$ is the only real root of the original cubic equation.

(c) That α is a root of the equation means that

$$\alpha^3 + \beta\alpha - \alpha = 0.$$

Hence, taking into account that $\alpha \neq 0$, it must be the case that $\beta = 1 - \alpha^2$. This answers the first part of the question. To show that α is the only real root provided $|\alpha| < 2$, we proceed with long division as above:

$$\begin{aligned} x^3 + \beta x - \alpha &= x^3 + \left(1 - \alpha^2\right) x - \alpha \\ &= (x - \alpha) x^2 + \alpha x^2 + \left(1 - \alpha^2\right) x - \alpha \\ &= (x - \alpha) x^2 + (x - \alpha) \alpha x + (x - \alpha) \\ &= (x - \alpha) \left(x^2 + \alpha x + 1\right). \end{aligned}$$

The latter will have α as its only real root *precisely when* the discriminant of $x^2 + \alpha x + 1$, $D = \alpha^2 - 4 \cdot 1 \cdot 1 < 0$, i.e. $\alpha^2 < 4$, or $|\alpha| < 2$. We have used here two facts:

- A quadratic polynomial has two real, one multiple real, or two complex roots, according as its discriminant D is positive, zero, or negative.
- The number α is never a root of $x^2 + \alpha x + 1$, so that even if $D = 0$, α is still not the only root of our cubic equation.

(d) The student's method is as follows: $x(x^2 + \beta) = \alpha$ implies that $x = \alpha$ or $x^2 + \beta = \alpha$. So to get two distinct roots, each distinct from α, we require that $\alpha > \beta$. (If $\alpha - \beta$ was negative, then clearly there would be no real x satisfying $x^2 = \alpha - \beta$. If $\alpha - \beta$ was zero, then the only solution to $x^2 = \alpha - \beta$ would be $x = 0$.) Now, using $\beta = 1 - \alpha^2$, this means $\alpha^2 + \alpha - 1 > 0$. The two solutions of the quadratic equation $\alpha^2 + \alpha - 1 = 0$ can be computed using the quadratic formula to be

$$\frac{-1 \pm \sqrt{5}}{2}.$$

Since the coefficient of α^2 in $\alpha^2 + \alpha - 1$ is greater than 0, the curve $y(\alpha) = \alpha^2 + \alpha - 1$ is only strictly positive for $\alpha < (-1 - \sqrt{5})/2$ or $\alpha > (-1 + \sqrt{5})/2$. This takes care of the student's requirement. At the same time, there should in fact be only one real solution to the equation, and this requires that $|\alpha| < 2$ (by part (c) of the question). It follows that the range of possible values for α is given by

$$\left(-2, \frac{-1 - \sqrt{5}}{2}\right) \cup \left(\frac{-1 + \sqrt{5}}{2}, 2\right).$$

That both these intervals are non-empty is easily checked.