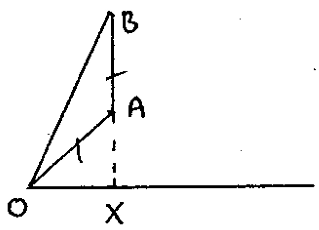
AEA 2003 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. Since we are asked to calculate the value $\tan(3\pi/8)$, it will certainly be agreeable to find a right-angled triangle with $3\pi/8$ for one of its angles. We can then use the formula $\tan(\alpha) = o/a$, where o (resp. a) is the length of the opposite (resp. adjacent) side to the acute angle $\alpha = 3\pi/8$. To this end, let the point X be the orthogonal projection of A onto the x-axis:



We shall show $\triangle OXB$ meets our needs. Firstly, since A has the same horizontal and vertical displacement from O, it must be the case that $\triangle OXA$ is an isosceles triangle with right angle $\angle OXA$. Secondly, we can compute using Pythagoras' theorem that

$$|OA| = \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1.$$

We also know that $|AB| = |b - a| = |j| = 1$. Hence $\triangle OAB$ is another isosceles triangle. These are key observations, which we note our sketch was helpful in identifying. (In general, when it comes to geometry problems, it can often be a good idea to spend some time on a good sketch and try and extract useful hypotheses/data therefrom.) Now, to identify the angles, we start by observing that

$$\angle OAX = \angle AOX = \frac{\pi - \pi/2}{2} = \frac{\pi}{4}$$

by the angle sum theorem for the isosceles $\triangle OXA$. Thus

$$\angle OAB = \pi - \angle OAX = \frac{3\pi}{4},$$

since the points X, A and B all lie on a common line (which is parallel to the y-axis). Another application of the angle sum theorem, this time for the isosceles $\triangle OAB$ yields

$$\angle BOA = \angle ABO = \frac{\pi - 3\pi/4}{2} = \frac{\pi}{8}.$$

Finally,

$$\angle XOB = \angle XOA + \angle AOB = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}.$$

We thus have our right-angled $\triangle OXB$, with right angle $\angle OXB$. We conclude that

$$\tan\left(\frac{3\pi}{8}\right) = \frac{|BX|}{|OX|} = \frac{1 + \frac{1}{\sqrt{2}}}{1/\sqrt{2}} = 1 + \sqrt{2}.$$

2. We first remark that we should consider $\theta \notin \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$ in order for $\tan(\theta)$ (or equivalently $\sec(\theta) = 1/\cos(\theta)$) to be well-defined. Now, since we are required to find the possible values of $v := \tan(\theta)$, rather than trying to find θ directly, let us endeavour to express the quantities appearing in the equality:

$$2\sin^2(\theta) - \sin(\theta)\sec(\theta) = 2\sin(2\theta) - 2 \tag{1}$$

in terms of v. It is clear, by definitions of trigonometric functions, that

$$\sin(\theta)\sec(\theta) = \sin(\theta)/\cos(\theta) = \tan(\theta) = v,$$

and so (1) can be written

$$2\sin^2(\theta) - v = 2\sin(2\theta) - 2.$$

Next, we divide by $\cos^2(\theta)$ (which we are assuming is non-zero) to obtain

$$2v^2 - \frac{1}{\cos^2(\theta)}v = \frac{2\sin(2\theta)}{\cos^2(\theta)} - \frac{2}{\cos^2(\theta)}.$$

Next, we use the double-angled formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to get rid of the $\sin(2\theta)$, finding

$$2v^2 - \frac{1}{\cos^2(\theta)}v = 4v - \frac{2}{\cos^2(\theta)}.$$

To convert the $1/\cos^2(\theta)$ terms, we use the formula $\sec^2(\theta) = 1 + \tan^2(\theta)$. This gives

$$2v^2 - (1 + v^2)v = 4v - 2(1 + v^2),$$

and a bit of tidying up yields

$$v^3 - 4v^2 + 5v - 2 = 0.$$

Now, this is a cubic equation in v, and we immediately observe that one of its solutions is given by $v_1 = 1$ (since the coefficients of the polynomial add to zero). Then we can perform polynomial long division:

$$\begin{aligned} v^3 - 4v^2 + 5v - 2 &= v^2(v-1) + v^2 - 4v^2 + 5v - 2 \\ &= v^2(v-1) - 3v(v-1) - 3v + 5v - 2 \\ &= (v^2 - 3v + 2)(v-1) \\ &= (v-1)(v-2)(v-1). \end{aligned}$$

It follows that the only other possible root is $v_2 = 2$. Moreover, since the range of the function tan is the whole of $\mathbb{R}$, there will indeed be θ for which the equality (1) holds and simultaneously $\tan(\theta) = v_i$, $i \in \{1, 2\}$, i.e. the sought-after values of $\tan(\theta)$ form precisely the set $\{1, 2\}$.

An alternative (and perhaps quicker solution) is as follows. First, using $\sec(\theta) = 1/\cos(\theta)$, we can write (1) as

$$2\sin^2(\theta) - \frac{\sin(\theta)}{\cos(\theta)} = 2\sin(2\theta) - 2.$$

Factoring out $\tan(\theta)$ on the left-hand side,

$$\frac{\sin(\theta)}{\cos(\theta)}(2\sin(\theta)\cos(\theta) - 1) = 2(\sin(2\theta) - 1).$$

We can then apply the double-angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to deduce

$$\tan(\theta)(\sin(2\theta) - 1) = 2(\sin(2\theta) - 1).$$

Moving everything across to one side, this is equivalent to

$$(\tan(\theta) - 2)(\sin(2\theta) - 1) = 0,$$

which holds whenever either $\tan(\theta) = 2$ or $\sin(2\theta) = 1$. Finally, $\sin(2\theta) = 1$ holds if and only if $2\theta = \frac{\pi}{2} + 2k\pi$ for some $k \in \mathbb{Z}$, and for these values of θ we have that $\tan(\theta) = 1$. The same conclusions as above follow.

3. We start by finding the coordinates of Q. From this, the area of the shaded region will follow quickly by performing a suitable integration.

   First, to obtain the equation for the tangent T to C at the point P, we calculate its slope s by differentiation:

$$s = \left.\frac{dy}{dx}\right|_{x=8} = \left.\frac{dy/dt}{dx/dt}\right|_{t=2} = \left.\frac{2t}{3t^2}\right|_{t=2} = \frac{1}{3}.$$

   Denoting by (x_P, y_P) the coordinates of P, it follows that the equation of T is $y - y_P = s(x - x_P)$, i.e. $y - 4 = \frac{1}{3}(x - 8)$ or $3y = x + 4$.

   Next, to find the coordinates of Q (equivalently the value of t in the parametrization of the curve C), we must determine where (other than the point P), T meets C. Plugging into the equation for T the parametrization of C, we have

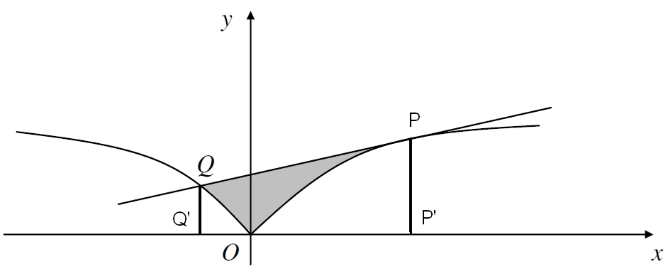
$$3t^2 = t^3 + 4. \tag{2}$$

   Now, since the point P lies on C and T, it must be the case that $t = 2$ solves (2). We can perform polynomial long division to find the other root(s):

$$\begin{aligned} t^3 - 3t^2 + 4 &= t^2(t-2) + 2t^2 - 3t^2 + 4 \\ &= t^2(t-2) - (t-2)(t+2) \\ &= (t-2)(t^2 - t - 2) \\ &= (t-2)^2(t+1). \end{aligned}$$

   Thus the only other root, and that corresponding to Q, is $t = -1$. Hence Q has coordinates $((-1)^3, (-1)^2) = (-1, 1)$.

   Finally, we obtain the area of the shaded region A by subtracting the area A_1 between the x-axis and the curve C (in the range $t \in [-1, 2]$) from the area A_2 of the trapezium $PQQ'P'$, as shown on the following figure:



The area A_1 follows by integration (we use a change of variables, but one need not (the alternative being to express $y(x)$ explicitly in terms of x, and then integrating directly.). In particular, A_1 is given by

$$\int_{x_Q}^{x_P} y dx = \int_{-1}^{2} y \frac{dx}{dt} dt = \int_{-1}^{2} (t^2)(3t^2) dt = 3\int_{-1}^{2} t^4 dt = \left[\frac{3}{5}t^5\right]_{t=-1}^{t=2} = \frac{3}{5}(32+1) = \frac{99}{5}.$$

On the other hand, we can compute A_2 directly:

$$A_2 = |P'Q'|\frac{|PP| + |QQ'|}{2} = (x_P - x_Q)\frac{y_P + y_Q}{2} = (8-(-1))\frac{4+1}{2} = \frac{45}{2}.$$

Hence we conclude that

$$A = A_2 - A_1 = \frac{225}{10} - \frac{198}{10} = 2.7.$$

4. (a) We first collect together all the terms of

$$\frac{Ax+B}{1+3x^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2}$$

over a common denominator, which gives

$$\frac{(Ax+B)(1-x)^2 + C(1-x)(1+3x^2) + D(1+3x^2)}{(1+3x^2)(1-x)^2}.$$

If this expression is to be equal to $f(x)$, we must have that

$$(A-3C)x^3 + (-2A+B+3C+3D)x^2 + (A-2B-C)x + (B+C+D) = 1-3x$$

(for every $x \neq 1$). Plainly then the coefficients before the respective terms $1 = x^0$, $x = x^1$, x^2 and x^3 on the left-hand and the right-hand side must be the same, i.e.

$$\begin{aligned} 0 &= A - 3C, & (3)\\ 0 &= -2A + B + 3C + 3D, & (4)\\ -3 &= A - 2B - C, & (5)\\ 1 &= B + C + D. & (6) \end{aligned}$$

Now, from (3), $A = 3C$, and then $3\times(6)+(5)-(4)$ yields $C = 0$. It follows that $A = 0$ too, and (5) gives $B = 3/2$. From (4) we have, finally, $B = -3D$, so that $D = -1/2$. (Of course, there are many other ways to go about solving the above equations.) To summarise, we have shown that, for $x \neq 1$,

$$f(x) = \frac{3}{2(1+3x^2)} - \frac{1}{2(1-x)^2}. \qquad (7)$$

(b) Recall the binomial series:

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,\ |z| < 1,\ \alpha \in \mathbb{R},$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

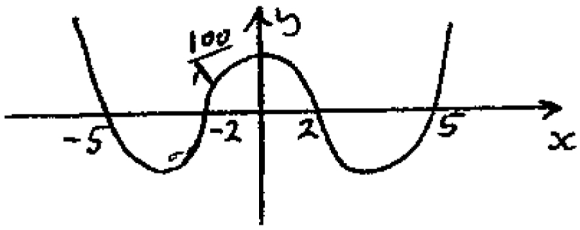
By applying this identity with $(\alpha, z) = (-1, 3x^2)$ and $(\alpha, z) = (-2, -x)$, we obtain from (7) that

$$\begin{aligned} f(x) &= \frac{3}{2}\left(1 + \frac{(-1)}{1!}3x^2 + \frac{(-1)(-2)}{2!}(3x^2)^2 + \cdots\right) \\ &\quad -\frac{1}{2}\left(1 + \frac{(-2)}{1!}(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + 4x^3 + 5x^4 + \cdots\right) \\ &= \frac{3}{2} - \frac{9}{2}x^2 + \frac{27}{2}x^4 + \cdots - \frac{1}{2} - x - \frac{3}{2}x^2 - 2x^3 - \frac{5}{2}x^4 + \cdots \\ &= 1 - x - 6x^2 - 2x^3 + 11x^4 + \cdots, \end{aligned}$$

for $|x| < 1$, as required.

(c) To find the equation of the tangent to f at $x = 0$, we need to compute $f(0)$ and $f'(0)$. This can be done by using the series expansion from (4b). In particular, we easily read off that $f(0) = 1$. Moreover, $f'(x) = -1 + 12x + \cdots$ for $|x| < 1$, and so $f'(0) = -1$. The equation of tangent at the point where $x = 0$, i.e. $y - f(0) = f'(0)(x - 0)$, is therefore $y = 1 - x$.

5. (a) Note that λ is simply a scaling factor for the y-coordinate. The crossing of the graph with the y-axis is obtained by computing $f(0) = 100/\lambda$. Moreover, the crossing points on the x-axis are given by solving the equation $f(x) = 0$ in x. Since $(x^2-4)(x^2-25) = (x-2)(x+2)(x-5)(x+5)$, we obtain the zeros ± 2 and ± 5. Further, f is an even function, with $\lim_{x\to\pm\infty} f(x) = +\infty$, positive on $(-\infty, -5) \cup (-2, 2) \cup (5, \infty)$ and negative on $(-5, -2) \cup (2, 5)$. We thus are able to sketch the graph as follows:



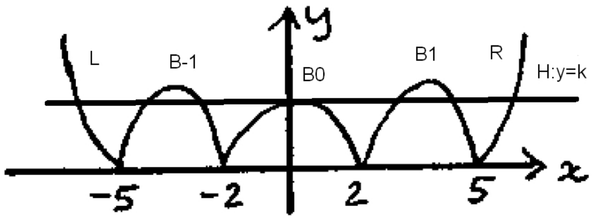
(b) To find the range of f, we need to find its minimum. To this end, we compute the derivative of f using the product rule $(gh)' = g'h + gh'$ with $g(x) = x^2 - 4$ and $h(x) = x^2 - 25$. In particular,

$$\lambda f'(x) = 2x\left((x^2 - 25) + (x^2 - 4)\right) = 2x(2x^2 - 29).$$

The roots of $f'(x) = 0$ are then 0 and $\pm\sqrt{29/2}$. We recognize the latter two are the x-coordinates of the global minima of f (as shown in our sketch above). (We can convince ourselves that these are indeed coordinates of global minima by observing that f' is negative (resp. positive, negative, positive) on $(-\infty, -\sqrt{29/2})$ (resp. $(-\sqrt{29/2}, 0)$, $(0, \sqrt{29/2})$, $(\sqrt{29/2}, +\infty)$).) By the continuity of f, and the previously made observation that $\lim_{x\to\pm\infty} f(x) = +\infty$, we obtain the range of f as being

$$\left[f\left(\sqrt{\frac{29}{2}}\right), \infty\right) = \left[-\frac{441}{4\lambda}, \infty\right).$$

(c) The figure below shows a sketch of the graph of $|f|$. Recall from (5b) that $a_1 := |f(\pm\sqrt{29/2})| = 110.25/\lambda$, whereas $a_0 = f(0) = 100/\lambda$. In particular, note that we always have $a_0 < a_1$.



On the figure, we have superimposed a line H with equation $y = k$. In this particular configuration H crosses L and R, B_{-1} and B_1 and just touches B_0 for a total of $k = 7$ roots to the equation $|f| = k$. As we shall see, this situation would require λ not to be a positive integer, and so in fact does not contribute to the integer solution set for (k, λ).

From the figure, we identify the following possible values of k, according to which of L, R, B_{-1}, B_0 and B_1 the horizontal line H crosses/touches:

(i) $k = 2$: H crosses L and R. This condition is equivalent to $a_1 < k$, i.e. $\lambda > 110.25/2$. Since λ is required to be a positive integer, we have the pairs $(k, \lambda) \in \{(2, 56), (2, 57), \ldots\}$.

(ii) $k = 4$: H crosses L and R, just touches B_{-1} and B_1. Equivalently $a_1 = k$, i.e. $4 = 110.25/\lambda$, which does not yield an integer-valued solution for λ.

(iii) $k = 6$: H crosses L, R, B_{-1} and B_1. Equivalently $a_1 > k > a_0$, i.e. $110.25/\lambda > 6 > 100/\lambda$. This gives that $\lambda \in \{17, 18\}$. So here we get the pairs $(k, \lambda) \in \{(6, 17), (6, 18)\}$.

(iv) $k = 7$: H crosses L and R, B_{-1} and B_1, touches B_0. Equivalently $k = a_0$, i.e. $7 = 100/\lambda$, which again does not yield an integer-valued solution for λ.

(v) $k = 8$: H crosses L, R, B_{-1}, B_0, B_1. Equivalently $k < a_0$, i.e. $\lambda < 100/8$. Thus we get the pairs $(k, \lambda) \in \{(8, 1), \ldots, (8, 12)\}$.

In conclusion, the sought-after pairs of (k, λ) constitute the set

$$\{(2, n) : n \geq 56\} \cup \{(6, 17), (6, 18)\} \cup \{(8, n) : 1 \leq n \leq 12.\}.$$

6. (a) Since both the left-hand side $L = \sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}$ and the right-hand side $R = \sqrt{2}$ of the equality we are trying to establish are positive and real, to show that $L = R$, it will be enough to establish that $L^2 = R^2$. Now, using the relations $(a-b)^2 = a^2 - 2ab + b^2$ and $(a+b)(a-b) = a^2 - b^2$,

$$\begin{aligned} L^2 &= \left(\sqrt{2+\sqrt{3}}\right)^2 - 2\sqrt{(2+\sqrt{3})(2-\sqrt{3})} + \left(\sqrt{2-\sqrt{3}}\right)^2 \\ &= 2+\sqrt{3} - 2\sqrt{4-3} + 2 - \sqrt{3} \\ &= 2 \\ &= R^2, \end{aligned}$$

as required.

(b) By part (6a), we are being asked to show that

$$\log_{\frac{1}{8}}\left(\sqrt{2}\right) = -\frac{1}{6}.$$

The definition of the logarithm is that $\log_a x = b$ if and only if $x = a^b$ (where $a, x \in \mathbb{R}^+$, $b \in \mathbb{R}$). Hence, the above statement is equivalent to

$$\sqrt{2} = (1/8)^{-1/6}.$$

Now, the right-hand side of this satisfies

$$(1/8)^{-1/6} = ((1/2)^3)^{-1/6} = (1/2)^{-1/2} = 2^{1/2} = \sqrt{2},$$

which confirms the desired result. (Note that we have used the elementary relations $(a^x)^y = a^{xy}$ and $a^{-x} = 1/a^x$ for $a \in \mathbb{R}^+$, $x, y \in \mathbb{R}$.)

(c) Again applying the definition of the logarithm, we are looking for pairs of integers (a, n) such that

$$\sqrt{a+\sqrt{15}} - \sqrt{a-\sqrt{15}} = (1/n)^{-1/2} = \sqrt{n}.$$

We remark that is necessary to have $a \geq \sqrt{15}$ and $n \geq 1$. Both sides of this equation are then real and positive, hence squaring yields the equivalent demand

$$a + \sqrt{15} - 2\sqrt{a^2-15} + a - \sqrt{15} = n,$$

which after cancellation becomes

$$2(a - \sqrt{a^2-15}) = n.$$

Hence, we see that n must be even. We can thus divide by 2 to obtain

$$a - \frac{n}{2} = \sqrt{a^2-15}.$$

This implies that $b := a - (n/2)$ is greater than or equal to 0. As a result, the previous equation is equivalent to

$$b^2 = a^2 - 15,$$

which yields

$$(a-b)(a+b) = 15.$$

It is now essential to observe that a and b are both positive integers. In particular $a + b \geq a - b$. Further to this, 15 factorizes into a product of two integers in nondecreasing order in precisely two ways: $15 = 1 \cdot 15$ and $15 = 3 \cdot 5$. It follows that we can have:

(i) either $a - b = 1$ and $a + b = 15$, i.e. $a = 8$ and $b = 7$, hence $n = 2(a - b) = 2$;

(ii) or $a - b = 3$ and $a + b = 5$, i.e. $a = 4$ and $b = 1$, hence $n = 2(a - b) = 6$.

The sought-after pairs of integers (a, n) are therefore seen to be $(8, 2)$ and $(4, 6)$.

7. (a) The points where C crosses the x-axis are obtained by solving $e^{-x}\sin(x) = 0$. Since $e^{-x} > 0$ for all x, the only solutions of this are those for which $\sin(x) = 0$, i.e. $x = k\pi$, $k \in \mathbb{Z}$. Thus the coordinates of P, Q and R are $(\pi, 0)$, $(2\pi, 0)$ and $(3\pi, 0)$, respectively.

(b) The key here is to perform integration by parts ($\int u dv = uv - \int v du$) *twice*. First (with $u = e^{-x}$, $v = -\cos(x)$, $du = -e^{-x}dx$, $dv = \sin(x)dx$):

$$\begin{aligned} I &:= \int e^{-x}\sin(x)dx = e^{-x}(-\cos(x)) - \int(-\cos(x))(-e^{-x})dx \\ &= -e^{-x}\cos(x) - \int e^{-x}\cos(x)dx. \end{aligned}$$

Now do it again (but with $u = e^{-x}$, $v = \sin(x)$, $du = -e^{-x}dx$, $dv = \cos(x)dx$), to get:

$$I = -e^{-x}\cos(x) - \left(e^{-x}\sin(x) - \int \sin(x)(-e^{-x})dx\right).$$

On the right-hand side we recognize I (modulo an additive constant $2C$), from which we conclude

$$2I = -e^{-x}\left(\sin(x) + \cos(x)\right) + 2C,$$

i.e.

$$I = -\frac{1}{2}e^{-x}\left(\sin(x) + \cos(x)\right) + C.$$

(c) To find the area of A_n, by the findings of (7a), we need to compute:

$$A_n = \int_{2(n-1)\pi}^{(2n-1)\pi} e^{-x}\sin(x)dx.$$

Using the formula from (7b), we have:

$$\begin{aligned} A_n &= \left[-\frac{1}{2}\left(\sin(x) + \cos(x)\right)\right]_{2(n-1)\pi}^{(2n-1)\pi} \\ &= \frac{1}{2}\left((0+1)e^{-2(n-1)\pi} - (0 + (-1))e^{-(2n-1)\pi}\right) \\ &= \frac{1}{2}e^{-2n\pi}e^{\pi}\left(e^{\pi} + 1\right). \end{aligned}$$

(d) It is clear from the final expression for A_n in (7c) that $\sum_{i=1}^{\infty} A_i$ is a geometric series of the form $a + ar + ar^2 + \cdots$ with $a = (1 + e^{-\pi})/2$ and $r = e^{-2\pi}$. Moreover, $|r| < 1$. Thus its sum is equal to:

$$S_\infty = \frac{a}{1-r} = \frac{1+e^{-\pi}}{2(1-e^{-2\pi})} = \frac{1+e^{-\pi}}{2(1-e^{-\pi})(1+e^{-\pi})} = \frac{1}{2(1-e^{-\pi})} = \frac{e^{\pi}}{2(e^{\pi}-1)},$$

where in the third to last equality we have used the formula $1 - x^2 = (1-x)(1+x)$ with $x = e^{-\pi}$.

(e) Denote

$$\begin{aligned} I &= \int_0^{\infty} e^{-x}\sin(x)dx, \\ S &= \int_0^{\infty} |e^{-x}\sin(x)|dx, \\ S_A &= \sum_{n=1}^{\infty} A_n, \\ S_B &= \sum_{n=1}^{\infty} B_n, \end{aligned}$$

where $B_1, B_2, \ldots$ represent the (negative) areas between the x-axis and successive portions of C where y is negative. We then have that

$$S = S_A - S_B, \qquad I = S_A + S_B.$$

We have already identified $S_A = e^{\pi}/2(e^{\pi} - 1)$ in (7d) and we are given $I = 1/2$. Hence $S_B = I - S_A = -1/(2(e^{\pi} - 1))$. Finally,

$$S = S_A - S_B = \frac{e^{\pi}}{2(e^{\pi}-1)} + \frac{1}{2(e^{\pi}-1)} = \frac{e^{\pi}+1}{2(e^{\pi}-1)}.$$