# Mathematics Techniques - EXTRAS

During the first week of term you will be examined on the basic mathematical techniques which are necessary for your degree, Differentiation, Integration and Trigonometry.

This file contains additional practice questions on both those topics and others that we feel are important for you to master before arriving.

The reference letters, e.g. [BC], refer to the list of recommended text books in the letter attached to the other file (sample questions for the Diagnostic Test).

## 1. Arithmetic and Algebraic Manipulation [BC chapter 1, ST chapter 0]

Arithmetic and algebraic manipulation is *not* included in the written examination

**Question 1** When $a$ and $b$ have the values stated below, find integers $q$ and $r$ where $0 \leq r < |b|$, and

$$a = qb + r$$

That is, find the quotient $q$ and remainder $r$ when the integer $b$ is divided into the integer $a$.

(a) $a = 100,\ b = 31$ (b) $a = 7,\ b = -2$ (c) $a = 1054,\ b = 1055$

(d) $a = -117,\ b = -29$ (e) $a = 126873,\ b = 127$ (f) $a = 10,\ b = 0$

**Question 2** **(i)** Factorize the following integers into a product of primes (a prime number $p$ has only the divisors 1 and $p$ ; for example $2, 3, 5, 7$, and 11 are all prime numbers):

(a) 42 (b) 63 (c) 385 (d) 1188 (e) 132461 (f) 9009

**(ii)** Using part (i), find the highest common factor and least common multiple of each of the following pairs of integers:

(a) 42, 63 (b) 385, 1188 (c) 132461, 9009

**Question 3** (BC $1d, e, f, g, k$; ST $0B$) Expand and simplify where possible

(a) $(2a-b)(3a-2b)$ (b) $(x-a)(x+a)$ (c) $(x+4)^2$ (d) $(2x-5y)^2$

(e) $5x(x^2-3x-4)-2(x+3)^2$ (f) $(x+1)^3$ (g) $(a+b)^4$ (h) $(a+b+c)^2$

(i) $(2x^2y-1)(2x^2y+1)$ (j) $x(x+1)(x+2)^3$ (k) $(a+1)^5$ (l) $(x+2x)^6$

**Question 4** (BC $1h, i, j$; ST $0B$) Factorize the following expressions

(a) $16x^2+24xy$ (b) $ax+3x+2a+6$ (c) $xa-2xb+ya-2yb$

(d) $x^2+8x+15$ (e) $x^2+2ax+a^2$ (f) $6x^2+x-12$

(g) $x^2-y^2$ (h) $x^4+4x^3+6x^2+4x+1$ (i) $x^2-25$

**Question 5** (BC $2a, b$) Simplify the following

(a) $\dfrac{x-2}{4x-8}$ (b) $\dfrac{x^2+xy}{xy+y^2}$ (c) $\dfrac{x^2-x-6}{2x^2-5x-3}$ (d) $\dfrac{(x-2)(x+2)}{x^2+x-2}$ (e) $(x+1)\times\dfrac{1}{(x^2-1)}$

**Question 6** (BC 2c, d; ST 0B; MK 1c) Express each of the following as a single fraction, simplifying where possible

(a) $\dfrac{1}{a}-\dfrac{1}{b}$ (b) $3x+\dfrac{1}{4x}$ (c) $1+\dfrac{1}{a}+\dfrac{1}{a+1}$

(d) $\dfrac{1}{x^2+2x+1}+\dfrac{1}{x+1}$ (e) $\dfrac{t+2}{(t+1)^2}-\dfrac{1}{t}$ (f) $\dfrac{1}{2(x+4)}+\dfrac{3}{(x+4)^2}+\dfrac{1}{2}$

(g) $\dfrac{2(x+4)}{5}\times\dfrac{1}{(x^2-16)}$ (h) $\dfrac{2}{(x+3)(x+1)}+\dfrac{3}{2x-1}$ (i) $\dfrac{3}{x}-\dfrac{2}{x(x-1)}$

(j) $\dfrac{5}{3-x}+\dfrac{2+x}{21-4x-x^2}-\dfrac{2x-3}{2x^2+13x-7}$ (k) $\dfrac{1/x}{1/y}\times\dfrac{1/z}{1/x}$ (l) $\dfrac{1}{(x-h)^2}-\dfrac{1}{(x+h)^2}$

**Question 7** (BC 3a; ST 0C) Express in terms of the simplest possible surd (for example, $\sqrt{8}=2\sqrt{2}$ and $\sqrt{45}=3\sqrt{5}$ etc)

(a) $\sqrt{12}$ (b) $\sqrt{32}$ (c) $\sqrt{200}$ (d) $\sqrt{288}$

(e) $\sqrt{8}+\sqrt{18}-2\sqrt{2}$ (f) $\sqrt{512}+\sqrt{128}+\sqrt{32}$ (g) $\sqrt{1000}-\sqrt{40}-\sqrt{90}$ (h) $\sqrt{1210}$

**Question 8** (BC 3b; ST 0C) Express each in the form $a+b\sqrt{c}$

(a) $\sqrt{3}(2-\sqrt{3})$ (b) $(\sqrt{3}+2)(\sqrt{3}+5)$ (c) $(4+\sqrt{7})(4-\sqrt{7})$

(d) $(\sqrt{6}-2)^2$ (e) $(3\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$ (f) $(1+\sqrt{2})^3$

**Question 9** (BC 3c; ST 0C) Express the following as fractions with integer denominators

(a) $\dfrac{3}{\sqrt{2}}$ (b) $\dfrac{\sqrt{5}}{\sqrt{10}}$ (c) $\dfrac{1}{\sqrt{2}-1}$ (d) $\dfrac{3}{\sqrt{3}-\sqrt{2}}$ (e) $\dfrac{4-\sqrt{3}}{3-\sqrt{3}}$ (f) $\dfrac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{3}}$

**Question 10** (BC 3d; ST 5A) Simplify

(a) $(a^2)^3$ (b) $(\sqrt{a})^3$ (c) $\dfrac{2^4}{2^2\times 4^3}$ (d) $(3^3)^{1/2}\times 9^{1/4}$ (e) $\dfrac{p^{1/2}\times p^{-3/4}}{p^{-1/4}}$ (f) $\dfrac{(16)^{5/4}}{8^{4/3}}$

(g) $\dfrac{x^{p+(1/2)q}\times y^{2p-q}}{(xy^2)^p\times\sqrt{x^q}}$ (h) $\dfrac{\sqrt{xy}\times x^{1/3}\times 2y^{1/4}}{(x^{10}y^9)^{1/12}}$ (i) $\dfrac{x^2(x^2+1)^{-1/2}-(x^2+1)^{1/2}}{x^2}$

**Question 11** (BC 3d; ST 5A) Find the values of

(a) $3^3$ (b) $27^{2/3}$ (c) $\left(\dfrac{1}{3}\right)^{-1}$ (d) $\left(\dfrac{100}{9}\right)^{0}$ (e) $\dfrac{2^{-1}}{3^{-2}}$

(f) $(0.64)^{-1/2}$ (g) $(2\frac{1}{4})^{-1/2}$ (h) $(3\frac{3}{8})^{-2/3}$ (i) $\dfrac{9^{1/3}\times 27^{-1/2}}{3^{-1/6}\times 3^{-2/3}}$ (j) $\dfrac{1}{3^{-1}}$

**Question 12** (BC 3f; ST 5C; MK 3A)

(i) Evaluate the following

$$\text{(a) } \log_{121} 11 \quad \text{(b) } \log_{1/2} 4 \quad \text{(c) } -\log_2\left(\tfrac{1}{8}\right) \quad \text{(d) } \log_{27} 3 \quad \text{(e) } \log_5\left(\tfrac{1}{125}\right)$$

(ii) Find the value of $x$ in the following

$$\text{(a) } \log_x 16 = 0.5 \quad \text{(b) } \log_8 16 = x \quad \text{(c) } -\log_{16} x = 0.75$$

In questions 13 and 14 the logarithms are taken to an *arbitrary* base

**Question 13** (BC 3$g$; ST 5$C$; MK 3$A$) Express in terms of $\log p$, $\log q$, and $\log r$

$$\text{(a) } \log(pqr) \quad \text{(b) } \log(p^2q^3/r) \quad \text{(c) } \log(p^n q^m) \quad \text{(d) } \log\left(p\sqrt{\frac{q}{r^3}}\right) \quad \text{(e) } \log\left(\frac{pq}{\sqrt{r^3}}\right)$$

**Question 14** (BC 3$g$,mix3; ST 5$C$; MK 3$A$) Express as single logarithms

(a) $3\log q + 4\log p$ (b) $\log p + 2\log q - 3\log r$ (c) $\log(1/p) + \log 1$

(d) $2\log 8 - \log 5 + 2\log 10$ (e) $\log(x+1) - \log 2$ (f) $4\log x - \log(x^2 + x^3)$

**Question 15** (ST 5$C$; MK 3$B$) Solve the equations

$$\text{(a) } 4^{-x} = 0.125 \quad \text{(b) } 4^{2x} + 4^{x+2} = 80 \quad \text{(c) } 3^{2x+1} = 3^x + 24 \quad \text{(d) } 3^{2x+1} + 3 = 10 \times 3^x$$

**Question 16** (BC 4$c$) Solve the following equations by completing the square, giving the solutions in surd form

$$\text{(a) } x^2 + 8x - 1 = 0 \quad \text{(b) } x + 4 = \frac{2}{x} \quad \text{(c) } 4 + 3x = 2x^2$$

**Question 17** Use the method of completing the square to show that the solutions of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with real roots provided $b^2 \geq 4ac$.

**Question 18** Solve the following equations for real $x$

(a) $x^3 - 5x^2 + 6x = 0$ (b) $x^4 - 5x^2 - 36 = 0$

(c) $x^5 + k^2x = 0$ (d) $x^6 - 7k^3x^3 - 8k^6 = 0$

**Question 19** (BC 4$e$, $f$; ST 5$C$) Solve the following simultaneous equations

(a) $7x + 4y = 10$, $5x + 3y = 7$

(b) $x + 2y = 4$, $x + 3z = 5$, $2y - z = 1$

(c) $y = 3 + z$, $x - 2y + z = -4$, $x - 11 = -2y$

(d) $x + y + z = 0$, $2x + 2y + z = 0$, $3x + y + 5z = 0$

(e) $y = 4x^2$, $y + 2x = 2$

(f) $y^2 = 4x + 1$, $y = x + 1$

(g) $x + 3y = 0$, $2x + 3xy = 1$

(h) $2\log_x y = 1$, $xy = 64$

(i) $5^{x+2} + 7^{y+1} = 3468$, $7^y = 5^x - 76$

(j) $\log_e 6 + \log_e(x - 3) = 2\log_e y$, $2y - x = 3$

## 2. Differentiation [BC chapters 14,15,23, ST chapters 10,15 MK chapter 8]

**Question 1** (BC 14*d*, *e*, 22*a*, *b*, *c*,mix22; ST 10*B*; MK 8*A*) Differentiate with respect to $x$:

(a) $16$ (b) $x$ (c) $x^7$ (d) $4x^3+5x^2-9$

(e) $x^{1/2}$ (f) $x^{5/2}-4x^3$ (g) $\dfrac{5}{x}-\dfrac{2}{x^7}$ (h) $(x+2)^2$

(i) $(x-4)^{-2}$ (j) $\dfrac{1}{x-7}$ (k) $(x+1)^3(x+4)^4$ (l) $\dfrac{2+x^2}{1+x}$

(m) $x(x+1)^{1/2}(x-2)^2$ (n) $(x^2-1/x)^{-3}$ (o) $\sqrt{x^3}+\dfrac{1}{\sqrt{x}}$ (p) $\dfrac{4}{x}+\dfrac{5}{x^2}-\dfrac{6}{x^3}$

(q) $(a^2+x^2)^{1/n}$ (r) $\sqrt{(1-x)/(1+x)}$ (s) $\sqrt{x+\sqrt{x+\sqrt{x}}}$ (t) $x\sqrt{x}-x^2\sqrt{x}$

**Question 2** (BC 14*f*, 15*b*: ST 10*C*) (a) Find the gradient of $y=2x^2-x+4/x$ at $(2,8)$.

(b) Find the points on the curve $y=x^3+3x^2-5x-10$ where the gradient is 4.

(c) Find the coordinates and nature of any turning points on the curve $y=36x-3x^2-2x^3$.

(d) Find the minimum and maximum values of $36x-3x^2-2x^3$ in the range $-5\le x\le 5$.

(Hint: use part (c); check values at endpoints.)

**Question 3** (BC 23*e*, 25*c*, *d*, *e*,mix25; ST 15*F*; MK 8*B*) Differentiate the following functions with respect to $x$ (here log denotes natural logarithms $\ln=\log_e$):

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $(\cos x)^3$

(e) $\cos(x^3)$ (f) $\sin(2x)$ (g) $\sin^2(2x)$ (h) $x\sin(x^{1/2})$

(i) $e^x$ (j) $\dfrac{3e^x-1}{3e^x+1}$ (k) $xe^{x^2}$ (l) $e^{x^2-3x}$

(m) $\log x$ (n) $e^{\log x}$ (o) $\dfrac{\log(1-x^2)}{x^3}$ (p) $\dfrac{x^2}{\cos x}$

(q) $\log(x^{1/2}+x^{-1/2})$ (r) $\log(\sin x)$ (s) $\dfrac{\sin x}{e^{2x}}$ (t) $\sin^n x$

(u) $\dfrac{1}{\log x}$ (v) $\log(e^x+5)$ (w) $\dfrac{\log(2x-5)}{\cos^2(3x)}$ (x) $x^3e^x\sin(x/2)$

**Question 4** (a) If $y=\sin x-x\cos x$ show $\dfrac{dy}{dx}\ge 0$ for all $x$ in the interval $0\le x\le\dfrac{\pi}{2}$.

(b) Find the maximum value of $\sin x-x\cos x$ in the interval $0\le x\le\dfrac{\pi}{2}$.

**Question 5** (i) Let $f(x)$ be a function of $x$. Use the product rule to show that

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{1}{(f(x))^2}\frac{d}{dx}(f(x)),$$

and use this to obtain the quotient rule.

(ii) Differentiate the following functions with respect to $x$:

$$\text{(a) } \operatorname{cosec} x = \frac{1}{\sin x} \quad \text{(b) } \sec x = \frac{1}{\cos x} \quad \text{(c) } \cot x = \frac{1}{\tan x} \quad \text{(d) } \frac{1}{\log x}$$

**Question 6** Differentiate the function $f(x) = \sin^2 x + \cos^2 x$ with respect to $x$. Hence show that $\sin^2 x + \cos^2 x = 1$. (Hint: show that $f(x)$ is constant, then use $f(0) = 1$.)

**Question 7** Find $\frac{d^2}{dx^2}(f(x))$ and $\frac{d^3}{dx^3}(f(x))$ for each of the following functions $f(x)$:

$$\text{(a) } x^2 \quad \text{(b) } \sin^2 x \quad \text{(c) } \frac{e^x}{\log x} \quad \text{(d) } \frac{1}{x^{2n}} \quad \text{(e) } x^2(x-1)^3 \quad \text{(f) } \sqrt{2x^3}$$

**Question 8** Let $f, g, h$ be three functions of $x$. Assuming the product rule, find

$$\text{(a) } \frac{d^2}{dx^2}(fg) \quad \text{(b) } \frac{d^3}{dx^3}(fg) \quad \text{(c) } \frac{d}{dx}(fgh) \quad \text{(d) } \frac{d}{dx}(f^4) \quad \text{(e) } \frac{d^2}{dx^2}(f(g+h))$$

Can you guess what $\frac{d^n}{dx^n}(fg)$ is ?

**Question 9** (BC 15$b, c$; ST 10$D, F$; MK 8$C$) Find the stationary points of the following functions, and investigate their nature.

(a) $y = 3x - x^3$ (b) $y = x^2$ (c) $y = x^2(x^2 - 8)$

(d) $y = 4x^3 - x^4$ (e) $y = x^3 + 7$ (f) $y = x^6 - 6x^4$

**Question 10** Given that $y = (A + x)\cos x$, where $A$ is a constant, show that $\frac{d^2y}{dx^2} + y$ is independent of $A$.

## 3. Integration [BC chapters 30,31,33 ST chapters 12,15,20 MK chapter 9]

Throughout this section, log denotes natural logarithms ln or $\log_e$.

**Question 1** Write down the integrals with respect to $x$ of the following functions:

(a) 1 (b) $x$ (c) $x^n$ (d) $\sin x$

(e) $\sin(ax+b)$ (f) $\cos x$ (g) $\sin x + \cos x$ (h) $\cos(ax+b)$

(i) $(ax+b)^n$ ; $n \neq 1$ (j) $\dfrac{1}{ax+b}$ (k) $e^x$ (l) $e^{ax+b}$

**Question 2** (BC 31*e*, 33*a*; ST 20*A*, *C*; MK 9*C*) Integrate the following functions with respect to $x$, using the substitutions given.

(a) $\tan x$ ; put $u = \cos x$

(b) $\dfrac{\cos x}{4+\sin x}$ ; put $u = \sin x$

(c) $\sin^3 x \cos x$ ; put $u = \sin x$

(d) $\dfrac{1}{2\sqrt{x}(1+x)}$ ; put $x = u^2$

(e) $\dfrac{1}{1+x^2}$ ; put $x = \tan u$

(f) $x^3\sqrt{x^4-1}$ ; put $u = x^4$

(g) $x^2(9-x^2)^{1/2}$ ; put $x = 3\sin u$

(h) $\dfrac{(\log x)^2}{x}$ ; put $u = \log x$

(i) $\sin^3 x$ ; put $u = \cos x$ (hint : use $\sin^2 + \cos^2 x = 1$)

**Question 3** (BC 33*d*; ST 15*G*; MK 9*B*)

(a) Find $\int \cos^2 x dx$ (hint: consider $\cos(2x)$).

(b) Find $\int (1-2\sin x)^2 dx$ (hint: use (a)).

(c) Find $\int \sin^5 x dx$ (hint: write $\sin^5 x$ as $(\sin^2 x)^2 \sin x$, and use $\sin^2 x + \cos^2 x = 1$).

(d) Find $\int \tan^2 x dx$ (hint: use $\sec^2 x = \tan^2 x + 1$, and note that $\dfrac{d}{dx}(\tan x) = \sec^2 x$).

(e) Show that $\sin^2 x + 3\cos^2 x = 2 + \cos(2x)$. Hence evaluate $\displaystyle\int_{\pi/12}^{\pi/4} (\sin^2 x + 3\cos^2 x) dx$.

(f) Find $\int \sec x dx$.

(Hint: write $\sec x = \sec x \left(\dfrac{\sec x + \tan x}{\sec x + \tan x}\right) = \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$, and note that

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x.)$$

(g) Find $\int \sin x \cos(x/2) dx$ (hint: consider $\sin(A+B) + \sin(A-B)$).

**Question 4** (BC 31$g$; ST 20$B, C$; MK 9$D$) Integrate the following functions with respect to $x$, using integration by parts.

(a) $x \cos x$ (b) $x \sin x$ (c) $x \log x$ (d) $x^2 e^{-x}$ (e) $e^x(x+1)$ (f) $(\log x)^2$

(g) $\log x$ (h) $x^n \log x$ (i) $\dfrac{\log x}{x^3}$ (j) $e^x \sin x$ (k) $x^2 \cos^2 x$ (l) $x^3 \sin(x^2)$

**Question 5** (BC 30$b$, 31$f, h$; ST 12$B$, 15$G$, 19$B$; MK 9$D, E$) Evaluate the following definite integrals:

(a) $\displaystyle\int_3^5 \frac{4}{1-x}dx$ (b) $\displaystyle\int_0^2 (x-1)(x+1)^3 dx$ (c) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cot x}{\sqrt{\operatorname{cosec}^3 x}}dx$ ; put $\operatorname{cosec} x = u$

**Question 6** (a) If $y = x^2$, show by means of sketch graphs and *not* by evaluating the integrals, that $\displaystyle\int_0^1 y dx = 1 - \int_0^1 x dy$.

(b) Explain why $\displaystyle\int_{-5}^{5} \sin^{17} x dx = 0$, *without* evaluating the indefinite integral $\displaystyle\int \sin^{17} x dx$.

**Question 7** (BC 30$c, d, e$, 35$c$; ST 12$B$; MK 9$F$)

(a) Find the area of the region bounded by the curve $f(x) = \log x$, the $x$–axis and the lines $x = 2$ and $x = 5$. (Hint: draw a sketch.)

(b) The region $R$ is bounded by the curve $y = \cos^2(2x)$, the $x$–axis and the lines $x = 0$ and $x = \pi/6$. Draw a sketch of $R$, and find its area.

(c) Find the area enclosed by the $y$–axis, and the graphs $y = x^2 - 1$ and $y = \cos(\pi x/2)$.

## 4. Trigonometry [BC chapters 16,17,24, ST chapters 4,15 MK chapter 6]

**Question 1** (i) Convert the following angles from radians to degrees:

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{7\pi}{12}$ (e) $\frac{7\pi}{2}$ (f) $\pi$

(ii) Convert the following angles to radians, leaving $\pi$ in your answer:

(a) $45°$ (b) $1800°$ (c) $1080°$ (d) $37°30'$ (e) $0°$ (f) $-90°$

From now on, all angles are in radians.

**Question 2** (BC 16*a*, *b*; ST 0*E*; MK 6*A*) Write down the values of sin, cos, and tan of the following angles (you will be expected to know these by heart):

(a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$ (e) $-\pi$ (f) $2\pi$ (g) $-\frac{4\pi}{3}$ (h) $\frac{5\pi}{4}$

**Question 3** (BC 16*d*, *e*, *f*; ST 4*A*, *B*; MK 6.2) For each of the functions below, state whether it is *odd* ($f(x) = -f(-x)$ for all $x$), *even* ($f(x) = f(-x)$ for all $x$), or neither. Find the *period* (the smallest positive real number $\tau$ with $f(x+\tau) = f(x)$ for all $x$) in each case, and sketch the graph between 0 and $3\pi$.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$ (d) $y = -\cos x$

(e) $y = -\sin x$ (f) $y = 1 \cos x$ (g) $y = \sin(2x)$ (h) $y = \sin(x/2)$

(i) $y = \sin x - \cos x$ (j) $y = \sin(x + 1/2)$ (k) $y = \tan(x + \pi/6)$ (l) $y = |\sin x|$

**Question 4** (BC 17*b*; ST 4*C*; MK 6*A*) Find the values of $x$ from $\pi$ to $2\pi$, inclusive, which satisfy the following equations:

(a) $\sin x = -1/2$ (b) $\cos x = -1/2$ (c) $\tan x = 2\sin x$

(d) $\cos(x + \pi/3) = 1/2$ (e) $1/\sin x = 2$ (f) $\sin^2 x + \sin x \cos x = 0$

(g) $\sin^2 x = 1/4$ (h) $\tan^2 x = \tan x$ (i) $2\cos^2 x = \cos x$

**Question 5** If $s = \sin\theta$ and $c = \cos\theta$, simplify

(a) $\frac{1-s^2}{1-c^2}$ (b) $\frac{c}{s} + \frac{s}{c}$ (c) $((s+c)^2 + (s-c)^2)^2$ (d) $\frac{\sqrt{1+(s/c)^2}}{\sqrt{1-s^2}}$ (e) $\frac{c^4 - s^4}{c^2 - s^2}$

**Question 6** (BC 17*a*; ST 4*D*; MK 6*C*) Show that the following identities hold for all $x$:

(a) $\cos^4 x - \sin^4 x + 1 = 2\cos^2 x$ (b) $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

(c) $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\operatorname{cosec} x$ (d) $\operatorname{cosec} x - \sin x = \cot x \cos x$

(e) $(1 + \cot x - \operatorname{cosec} x)(1 + \tan x + \sec x) = 2$ (f) $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x$

(Note that $\operatorname{cosec} x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, and $\cot x = \frac{1}{\tan x}$.)

**Question 7** (BC 17*a*; ST 4*D*; MK 6*D*) Eliminate $\theta$ from the equations

(a) $x = 2\cos\theta$ and $y = 3\sin\theta$ (b) $x + y = \cos\theta + 1$ and $x - y = \sin\theta - 1$

(c) $x = a\,\text{cosec}\,\theta$ and $y = b\sec\theta$ (d) $x = \sec\theta + \tan\theta$ and $y = \sec\theta - \tan\theta$

**Question 8** (BC 18*b*, *c*,mix18; ST 4*E*, *F*; MK 6*D*, *E*)

(i) Write down $\sin(A+B)$ and $\cos(A+B)$ in terms of $\sin A, \sin B, \cos A$, and $\cos B$. Hence show that

(a) $\cos(3A) = 4\cos^3 A - 3\cos A$ (b) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(c) $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B}$ (d) $\tan(3A) - \tan A = 2\sin A \sec(3A)$

(e) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$ (f) $\tan(3A) = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

(g) $\cos A - \cos B = -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

(ii) Solve the following equations:

(a) $\cos x\cos(\pi/6) - \sin x \sin(\pi/6) = 1/2$ for $-\pi \le x \le \pi$.
(Hint: consider $\cos(A+B)$.)

(b) $\tan(2x) + \tan x = 0$ for $0 \le x \le 2\pi$. (Hint: express $\tan(2x)$ in terms of $\tan x$.)

(c) $2\tan x + \sin(2x)\sec x = 1 + \sec x$ for $0 \le x \le 2\pi$.

(d) $\sin\left(x + \dfrac{\pi}{6}\right) = 2\cos x$ for $0 \le x \le 2\pi$.

**Question 9** (a) If $A, B$, and $C$ are the angles of a triangle, show that:

$$\cos A + \cos(B+C) = 0,$$

and deduce that $\cos A + \cos(B-C) = 2\sin B \sin C$.

(b) Show that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B\tan C - \tan C \tan A - \tan A \tan B}.$$

Hence show that if $A, B, C$ are the angles of a triangle, then

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

## 5. Inequalities [BC Chapter 13, ST section 14.2]

To *solve* an inequality is to find all numbers which satisfy it. The collection of numbers which satisfy a given inequality is called its *solution set.* The following three rules allow you to simplify and solve complicated inequalities:

**(Rule 1)** The product $xy$ of two non-zero real numbers is positive if and only if $x$ and $y$ are either both positive or both negative. (Thus, their product is negative if and only if they have opposite signs.)

**(Rule 2)** If $x, y, z$ are real numbers, and if $x > y$, then $x + z > y + z$.

**(Rule 3)** If $x, y, z$ are real numbers, and if $x > y$, then

$$\begin{cases} xz > yz & \text{if } z > 0 \\ xz = yz = 0 & \text{if } z = 0 \\ xz < yz & \text{if } z < 0 \end{cases}$$

**Example** Solve the inequality $x < 6x - 7$.

**Solution**

$$\begin{aligned} x < 6x - 7 &\Leftrightarrow -5x < -7 \quad \text{(by rule 2)} \\ &\Leftrightarrow 5x > 7 \quad \text{(by rule 3)} \\ &\Leftrightarrow x > \frac{7}{5} \quad \text{(by rule 3)} \end{aligned}$$

So the solution set consists of all numbers, $x$, with $x > \frac{7}{5}$ which we write as $\{x \mid x > \frac{7}{5}\}$.

**Question 1** **(i)** If $pq > 0$, show that $p/q > 0$.

**(ii)** Suppose $a \geq b$ and $c \geq d$. In each of the cases below, either show that it is true, or find numbers $a, b, c, d$ which contradict it:

(a) $|a| \geq |b|$ (b) $a + c \geq b + d$ (c) $d - c \leq a - b$ (d) $ac \geq bd$

**Question 2** Is $999,999 \times 1,000,001$ greater than $10^{12}$ ? (No calculators!)

**Question 3** For how many integers $n$ is

(a) $-\frac{2}{5} < \frac{n}{17} < \frac{3}{5}$ (b) $-\frac{2}{5} < \frac{17}{n} < \frac{3}{5}$

(c) If $A, B, C$ are the angles of a triangle, show that

$$\sin(2A) + \sin(2B) + \sin(2C) = 4 \sin A \sin B \sin C.$$

**Question 10** (BC 24$a$, 41$a$; ST 15$B$)

(a) Express $a \cos x + b \sin x$ in the form $R \sin(x + \alpha)$.

(b) Find the maximum value of $2 \cos x + 2 \sin x$.

**Question 11** Find an expression for $\sin(A+B+C)$ in terms of the sines and cosines of the angles $A, B, C$. Do the same for $\cos(A + B + C)$.

**Question 4** (BC 13$a, b$, c,mix13, 40$e$; ST 5$D$, 14$B$) Solve the following inequalities for $x$

(a) $x > 5x - 2$ (b) $-3x > 6$ (c) $x > \frac{2}{x}$

(d) $\frac{1}{x} < x < 1$ (e) $x^2 > 1$ (f) $(x+3)(x-5) \geq 0$

(g) $(x+1)(x+4) \leq 4$ (h) $\frac{4x}{x+2} > 3$ (i) $(x+4)(x-1) < 6$

(j) $\frac{3}{x-1} > 2$ (k) $\frac{2x}{(x-4)^2} > 1$ (l) $\frac{x}{x-2} < \frac{x}{x-1}$

(m) $\frac{(x-1)(x-2)}{(x+1)(x-3)} < 0$ (n) $\frac{2x}{x-1} + \frac{x-5}{x-2} > 3$ (o) $-3 < 5 - 2x < 3$

**Question 5** Solve the following inequality for $x$

$$\frac{1}{4} \leq \frac{x}{x^2+4} \leq \frac{1}{4}$$

**Question 6** (BC 40$e$; ST 14$B$) Solve the following inequalities for $x$

(a) $|x-3| \leq 2$ (b) $|x-3| \geq 2$ (c) $|2x-3| > 5$ (d) $1 < |x| < 2$

(e) $x + |x| < 1$ (f) $|2|x| - 3| > \frac{1}{2}$ (g) $\frac{6}{|x|+1} < |x|$ (h) $6 - x > |3x-2|$

(i) $\left|\frac{x}{x-3}\right| < 2$ (j) $|2x+5| > |x+1|$ (k) $1 - |x| > x^2 - 1$ (l) $\frac{1}{|x|} > -x$

**Question 7** (a) For which values of $x$ is $\left|\frac{1}{1+2x}\right| = 1$ ?

(b) Solve the inequality $\left|\frac{1}{1+2x}\right| < 1.$

**Question 8** (BC 13$d$,mix13; ST 5$D$)

(a) Show that $a^2 + b^2 \geq 2ab$ for all real $a, b$. When does equality hold ?

(b) Suppose $a, b, c \geq 0$; use part (a) to show that

$$(a+b)(b+c)(c+a) \geq 8abc.$$

(c) Show that for any numbers $a, b, c$ the following inequality holds:

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

When does equality hold ?

(Hint: $2(a^2 + b^2 + c^2 - ab - bc - ca) = (a-b)^2 + (b-c)^2 + (c-a)^2$.)

(d) Show that $x^2 + 8x + 17$ is positive for all $x$. Hence solve the inequality

$$\frac{x-3}{x^2+8x+17} < 0$$

(e) Show that $x^2 + y^2 - 10y + 25 \geq 0$ for all real values of $x$ and $y$.

(f) Find the range of values $k$ can take if $x^2 + 4x + k - 1$ is positive for all real $x$.

**Question 9** (a) Show that if $m > 0$, then

$$m + \frac{4}{m^2} \geq 3.$$

When does equality hold ?

(Hint: factorize the expression $m^3 - 3m^2 + 4$.)

(b) Show that if $a > b > 0$, then

$$a + \frac{1}{(a-b)b} \geq 3.$$

When does equality hold?

(Hint: use $(a-2b)^2 \geq 0$, to show that $(a-b)b \leq a^2/4$; then use part (a).)

**Question 10** (BC 4$g$; ST 5$E$) The quadratic equations below are all of the form $ax^2 + bx + c = 0$. In each case find the discriminant $b^2 - 4ac$, and hence state whether the equation will have two real distinct roots, no real roots, or a repeated root.

(a) $x^2 + 7x = 2$ (b) $4x^2 - 12x + 9 = 0$ (c) $7x^2 + 11x + 5 = 0$

**Question 11** (ST 5$E$) Find the range of $f(x) = x^2 + 4x - 2$, by

(a) sketch (b) discriminant (c) completing the square

**Question 12** (a) Show that the product of two numbers $x$ and $y$ whose sum is constant is greatest when $x = y$.

(Hint: $x + y = c$ implies $xy = x(c-x) = -x^2 + cx$; the roots are $0, c$. As can be seen from the graph, the maximum value is attained at their midpoint; namely when $x = c/2$.)

(b) Show that if $x > 0$, $y > 0$ and $x + y = 1$, then

$$(1 + \frac{1}{x})(1 + \frac{1}{y}) \geq 9.$$

(Hint: show that the left-hand side is equal to $1 + 2/xy$, and use part (a) to show that $xy \leq 1/4$.)

**Question 13** Sketch the regions in the plane where

(a) $x + y > 0$

(b) $x^2 + y^2 < 4$

(c) $x + 2y - 5 < 0$

(d) $x + 2y - 5 < 0$ and $x + y > 0$

(e) $(x + 2y - 5)(x + y) < 0$

(f) $-1 < x + y < 1$ and $-1 < x - y < 1$

(g) $-1 < x + y < 1$ or $-1 < x - y < 1$

(h) $(x^2 + y^2 - 4)(x^2 + y^2 - 9) < 0$

## Harder Exercises

The exercises on this sheet are harder than those on the other worksheets. They are for you to do if
(a) you have satisfied yourself that you are really up to scratch, and to speed, on the basic skills tested on the other worksheets, and
(b) you enjoy doing them.
If you find them too hard, don't be demoralised: you may not yet be ready for some of the ideas that they involve. On the other hand, try not to be put off by the complexity of the expressions, and give yourself time for the ideas to sink in. There is certainly no need to do, or even attempt, all of them. Do as many as you enjoy, find interesting, and have time for.

### Arithmetic and Algebraic Manipulation

1. *Sums of powers*

(i) Show that $1+2+\ldots+n=\frac{1}{2}n(n+1)$. (Hint: write $1+2+\cdots+(n-1)+n=S$; then also (reversing the order) $n+(n-1)+\cdots+2+1=S$. Add together the two equalities, noting that $1+n=2+(n-1)=3+(n-2)=\cdots=n+1$.)

(ii) In this exercise we use the 'method of telescoping sums' to find an expression for the sum $1^2+2^2+\ldots+n^2$.
Show that for any $i$, $(i+1)^3-i^3=3i^2+3i+1$. Hence

$$\begin{aligned}
2^3-1^3 &= 3\times 1^2+3\times 1+1\\
3^3-2^3 &= 3\times 2^2+3\times 2+1\\
&\vdots\\
(n+1)^3-n^3 &= 3\times n^2+3\times n+1.
\end{aligned}$$

Add together all these equations to deduce that

$$(n+1)^3-1^3=3(1^2+2^2+\cdots+n^2)+3(1+2+\cdots+n)+n$$

and use this (and (i)) to derive an expression for $1^2+2^2+\ldots+n^2$. Note the way that the terms on the left cancel in pairs; this is what gives the method its name.

(iii) Use the method of telescoping sums to find an expression for $1^3 + 2^3 + \cdots + n^3$.

(iv) Find a (simple) expression for

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n\times(n+1)}.$$

(Hint: each term in the sum can be written as the difference of two simpler fractions).

2. *Symmetric Polynomials*

(i) A *polynomial* in the variables $x$ and $y$ is an expression like $x^2 + y^2$ or $x^3y - 5xy^4 + xy$.

A polynomial $p(x, y)$ is said to be *symmetric* if when $x$ and $y$ are interchanged, $p$ remains the same. For example, $p(x, y) = x+y, p(x, y) = xy$ and $p(x, y) = x^3 - x^2y - xy^2 + y^3$ are all symmetric, while $p(x, y) = x - y$ and $p(x, y) = x^3y + y^2$ are not. A famous theorem of Isaac Newton asserts that every symmetric polynomial can be rewritten as a polynomial in the 'elementary symmetric functions' $x+y$ and $xy$. For example, $x^2+y^2 = (x+y)^2-2xy$, and $x^3 + x^2y + xy^2 + y^3 = (x + y)^3 - 2xy(x + y)$. Find the corresponding expressions for the symmetric polynomials $x^3 + y^3$ and $x^4 + y^4$ in terms of $x + y$ and $xy$.

(ii) By analogy with the case of two variables, the polynomial $p(x, y, z)$ is said to be symmetric if it remains unchanged if $x$ and $y$ are exchanged, if $x$ and $z$ are exchanged and if $y$ and $z$ are exchanged. For example, $p(x, y, z) = x^4 + y^4 + z^4$ is symmetric, but $q(x, y, z) = x^2 + y^2 + z^3$ is not. For the case of 3 variables, Newton's theorem asserts that every symmetric polynomial of 3 variables can be written as a polynomial in the symmetric functions $x+y+z, xy+yz+zx$ and $xyz$. Express the polynomial $x^3+y^3+z^3$ in this form.

3. In this question all numbers $a, b, c, p, q, r$ etc. are positive. Show successively that

(i) $(a^p - b^p)(a^q - b^q) \geq 0$
(Hint: if $a \geq b$ then $a^r \geq b^r$ for any positive $r$; hence both factors on the left-hand side of the inequality have the same sign.)

(ii) $a^{p+q} + b^{p+q} \geq a^p b^q + a^q b^p$

(iii) $\frac{1}{2}(a^{p+q} + b^{p+q}) \geq \frac{1}{2}(a^p + b^p) \times \frac{1}{2}(a^q + b^q)$

(iv) $\frac{1}{2}(a^{p+q+r} + b^{p+q+r}) \geq \frac{1}{2}(a^p + b^p) \times \frac{1}{2}(a^q + b^q) \times \frac{1}{2}(a^r + b^r)$
(Hint: write $q+r$ in place of $q$ in (iii), and apply (iii) again to the right-hand side of the expression you get.)

(v) Convince yourself that

$$\frac{1}{2}(a^{p_1+\ldots+p_n} + b^{p_1+\ldots+p_n}) \geq \frac{1}{2}(a^{p_1} + b^{p_1}) \times \cdots \times \frac{1}{2}(a^{p_n} + b^{p_n})$$

for any positive numbers $p_1$ up to $p_n$. (Hint: first use the same idea as in (iv) to deduce the statement for $n = 4$ from the case where $n = 3$ (which you proved in (iii)); a formal proof of the general case would involve mathematical induction, which you may not have studied yet, so don't worry about this. Essentially, the idea is that the statement can be proved for any given $n$ by repeatedly using the same trick.)

(vi) $\frac{1}{2}(a^n + b^n) \geq (\frac{1}{2}(a+b))^n$
(Hint: special case of (v).)

(vi) $\frac{1}{3}(a^{p+q} + b^{p+q} + c^{p+q}) \geq \frac{1}{3}(a^p + b^p + c^p) \times \frac{1}{3}(a^q + b^q + c^q)$
(Hint: Use (ii) for each of the pairs $a$ and $b$, $b$ and $c$, $c$ and $a$, and then proceed as from (ii) to (iii).)

(viii) $\frac{1}{3}(a^{p+q+r} + b^{p+q+r} + c^{p+q+r}) \geq$
$\frac{1}{3}(a^p + b^p + c^p) \times \frac{1}{3}(a^q + b^q + c^q) \times \frac{1}{3}(a^r + b^r + c^r)$

(ix) $\frac{1}{3}(a^n + b^n + c^n) \geq (\frac{1}{3}(a+b+c))^n$

(x) $\frac{1}{r}(a_1{}^n + a_2{}^n + \cdots + a_r{}^n) \geq (\frac{1}{r}(a_1 + \cdots + a_r))^n$.

4. *Geometric mean* $\leq$ *Arithmetic mean*

Given positive numbers $a_1, \ldots, a_n$, their *geometric mean* $G$ is $(a_1 \ldots a_n)^{1/n}$, while their *arithmetic mean* $A$ is $(a_1 + \ldots + a_n)/n$. For example, the arithmetic mean of 4 and 9 is 6.5, while their geometric mean is 6. This exercise shows that in fact the geometric mean of the positive numbers $a_1, \cdots, a_n$ is always less than or equal to their arithmetic mean, with equality if and only if $a_1 = a_2 = \ldots = a_n$. The proof comes from the book 'Inequalities', by Hardy, Littlewood and Polya.

(i) For two numbers $a_1$ and $a_2$ (i.e. if $n = 2$), the argument is easy: since $G$ and $A$ are both non-negative, $G \leq A$ if and only if $G^2 \leq A^2$, and $A^2 - G^2$ is the square of a certain expression (which you should find).

(ii) Use (i) to show that

$$a_1 a_2 a_3 a_4 \leq \left(\frac{a_1 + a_2}{2}\right)^2 \left(\frac{a_3 + a_4}{2}\right)^2 \leq \left(\frac{a_1 + a_2 + a_3 + a_4}{4}\right)^4$$

and deduce that $G \leq A$ when $n = 4$.

(iii) Using the same idea as the step from (i) to (ii), deduce from (ii) that $G \leq A$ when $n = 8$. Convince yourself that (using the same idea again and again) the result holds when $n = 16,\ 32, \ldots$ i.e. when $n$ is a power of 2. Once again, a formal proof would require mathematical induction, as in question 1(v).

(iv) To prove that $G \leq A$ for any $n$, use a trick: given $a_1, \ldots, a_n$, let $N$ be some power of 2 which is greater than $n$. Define a new collection of numbers $b_1, \ldots, b_N$ by

$$b_1 = a_1, b_2 = a_2, \ldots, b_n = a_n,$$

$$b_{n+1} = b_{n+2} = \ldots = b_N = A$$

where $A$ is the arithmetic mean of $a_1, \ldots, a_n$. By (iii), the geometric mean of the $b$'s is less than or equal to their arithmetic mean. Show that this implies that

$$a_1 a_2 \ldots a_n A^{N-n} \leq \left(\frac{b_1 + b_2 + \ldots + b_N}{N}\right)^N.$$

(v) Show that the right hand side of the last inequality is equal to $A^N$, and deduce that $G \leq A$ for $a_1, \ldots, a_n$.

(vi) Show that if $a_1 = a_2 = \ldots = a_n$ then $G = A$. If on the contrary some two of the numbers (say $a_1$ and $a_2$) are distinct, show that replacing each of them by their arithmetic mean $(a_1 + a_2)/2$ leaves $A$ unchanged but increases $G$. Deduce that unless $a_1 = a_2 = \ldots = a_n$, $G < A$.

5. i) Show from first principles (e.g by completing the square) that the quadratic polynomial $ax^2 + bx + c$ has

- two real roots if $b^2 - 4ac > 0$
- one real root if $b^2 - 4ac = 0$
- no real roots if $b^2 - 4ac < 0$.

The quantity $b^2 - 4ac$ is known as the *discriminant* of the quadratic $ax^2 + bx + c$.

ii) In this exercise we derive an expression for the discriminant of a cubic polynomial, which will have an analogous property to the discriminant of a quadratic - its sign will tell us how many real roots the cubic has.

We begin with a very simple cubic: $f(x) = x^3 + px + q$.

(a) Show (by differentiating $f$ and by sketching its graph) that if $p \geq 0$ then $f(x)$ has exactly one real root.

(b) Now assume $p < 0$. Show that $f$ has one strict local maximum and one strict local minimum. Denote the value of $f$ at the local maximum and local minimum by $M$ and $m$ respectively. Find $M$ and $m$ in terms of the coefficients $p$ and $q$, and deduce that $M > m$. By looking at the graph of $f$, conclude that $f$ has

- 1 real root if $m > 0$ or $0 > M$
- 2 real roots if $m = 0$ or $M = 0$
- 3 real roots if $M > 0 > m$

(c) Now show that $p < 0$ and $M > 0 > m$ if and only if $0 > 4p^3 + 27q^2$, and go on to show that the value of $4p^3 + 27q^2$ determines which of the other cases occurs. This expression is thus the discriminant of the cubic $f(x)$.

(d) Given the cubic $g(x) = x^3 + bx^2 + cx + d$, show that the change of variable $x = \bar{x} - b/3$ brings it to the form $\bar{x}^3 + p\bar{x} + q$, where $p$ and $q$ depend on $b, c$ and $d$, and find $p$ and $q$ explicitly in terms of $b, c$ and $d$. Of course $x^3 + bx^2 + cx + d$ and $\bar{x}^3 + p\bar{x} + q$ have the same number of real roots, for if the former vanishes at some value $x$ then the latter vanishes at $x + b/3$, and vice versa. Use this fact to derive an expression for the discriminant of $x^3 + bx^2 + cx + d$ from what you have already done.

(e) Find the discriminant of the cubic $ax^3 + bx^2 + cx + d$.

(f) Decide how many real roots each of the following cubics has: $x^3 - x + 1$, $3x^3 - 2x + 2 + x - 1$, $x^3 - 5x^2 + 7$.

## DIFFERENTIATION

6. (i) Show that when $x$ is positive,

$$\frac{1}{x} > \log(1+x) - \log(x) > \frac{1}{1+x}.$$

(Hint: check that this inequality holds for some particular value of $x$, and then look at the derivatives of each of the three functions you are asked to compare, sketching a graph of all three to keep track of what you want to show.)

(ii) Deduce that $(1 + \frac{1}{x})^x$ increases, and $(1 + \frac{1}{x})^{x+1}$ decreases, as $x$ increases.

7. For each integer $n \geq 0$ we define a function $f_n(x)$ by the following formula:

$$f_n(x) = x^{2n+2} e^{1/x} \frac{d^{n+1}}{dx^{n+1}} e^{-1/x}$$

(where the last factor means the $(n+1)^{th}$ derivative of $e^{-1/x}$.)

(i) Show that $f_0$ is actually a polynomial in $x$.

(ii) Show that for $n \geq 1$,

$$f_n(x) = -(2nx - 1) f_{n-1}(x) + x^2 f'_{n-1}(x).$$

(Do not attempt to differentiate $e^{-1/x}$ $n+1$ times - the expression gets far too messy! Do use the fact that the derivative of the $n^{th}$ derivative of any function is just the $(n+1)^{th}$ derivative of that function!)

(iii) Deduce from (i) and (ii) that $f_1$, $f_2$ and indeed all the $f_n$ are actually polynomials.

## Integration

8. (i) By writing $\sin^n(x) = \sin^{n-1}(x)\sin(x)$, then using integration by parts (twice), show that

$$\int \sin^n(x)dx = -\cos(x)\sin^{n-1}(x)+(n-1)\int \sin^{n-2}(x)dx-(n-1)\int \sin^n(x)dx+C,$$

so that

$$n\int \sin^n(x)dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)dx + C.$$

A formula of this type is called a recurrence formula; it expresses something difficult ($\int \sin^n(x)dx$) in terms of something slightly easier ($\int \sin^{n-2}(x)dx$). Use this recurence formula to derive explicit formulae for $\int \sin^n(x)dx$ when $n = 3, 4$ and 6 (which do not involve any further integrals).

(ii) Find a recurrence formula for $\int \cos^n(x)dx$.

(iii) Find a recurrence formula for $\int e^x \sin^n(x)dx$. (You will have to be more persevering here!)

## Geography

9. A *map* is a division of the plane into regions (the 'countries'). A *colouring* of a map is a colouring of each country so that no two countries having a common boundary are coloured the same. Countries meeting only at a point may have the same colour. For example, if the plane is divided into four countries (the quadrants) by the two coordinate axes, then the first and third quadrant may be one colour and the second and fourth quadrants another.

A famous theorem, conjectured in 1852 and not proved until 1977, asserts that four colours suffice to colour any map. Of course, not every map requires as many as four; the example described above needs only two. Show that in fact any map formed like the one described above, by dividing up the plane by means of a finite collection of straight lines, needs only two colours. Clarification: the straight lines in question must be infinite in both directions; for example, the map formed by dividing the plane by the whole of the $x$-axis and the positive half of the $y$-axis needs three colours, since each two countries have a common boundary.

Hint: this exercise can be done using the ideas you dealt with in question 13 on the Inequalities worksheet.

# Mathematical Techniques : Inequalities SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

**Section A.** State whether the following inequalities are true or false for all real numbers $a > b > 0$ and $c < 0$.

1. $\dfrac{c}{(a-b)} < 0$ (2)
2. $\dfrac{a}{c} < \dfrac{b}{c}$ (2)
3. $|c| > a - b$ (2)
4. $a^2bc^3 < ab^2c^3$ (2)

**Section B.** Determine the set of values of $x$ for which the following inequalities hold.

5. $x^2 - 6 > x$ (2)
6. $(x-1)(x+2) > (x+1)(x-2)$ (2)
7. $\dfrac{x^4+4}{5} \geq x^2$ (3)
8. $x - 4 < x(x-4) < 5$ (3)

**Section C.** Determine the set of values of $x$ for which the following inequalities hold.

9. $\left|\dfrac{x+3}{x+2}\right| < 1$ (3)
10. $|x^2 - 2| \leq 1$ (3)
11. $|x-1| > 1 + |1+x|$ (3)
12. $\dfrac{|x|-3}{-x^2-1} \geq 1$ (3)

# Inequalities: solutions

**(A)**

1. TRUE
2. TRUE
3. FALSE if $c=-1, a=10, b=1$ will not work.
4. TRUE $a>b \Rightarrow a(ab)>b(ab)$
   $\Rightarrow a^2b > ab^2$
   $\Rightarrow a^2bc^3 < ab^2c^3$ since $c^3<0$

**(B)** 5. $x^2-6>x$ means $x^2-x-6>0$ or $(x-3)(x+2)>0$

so either both brackets positive, $x>3$
or both brackets negative, $x<-2$.

so $\underline{x>3 \text{ or } x<-2}$.

6. $x^2+x-3>x^2-x-3$ so $2x>0$ so $\underline{x>0}$.

7. $\frac{x^4+4}{5} \geq x^2 \Rightarrow x^4-5x^2+4 \geq 0$ or $(x^2-4)(x^2-1) \geq 0$

by same reasoning as Question 5

$x \geq 2$ or $x \leq 2$ or $-1 \leq x \leq 1$. (Draw a graph?)

8. $x-4<x(x-4) \Rightarrow x>4$ or $x<1$

$x^2-4x<5 \Rightarrow (x-5)(x+1)<0$ so $-1<x<5$

Combine the two: $4<x<5$ or $-1<x<1$.

**(C)** 9. $\left|\frac{x+3}{x+2}\right|<1 \Rightarrow -1<\frac{x+3}{x+2}<1$

note special case of $x=-2$ so if

$\underline{x>-2}$ need $-x-2<x+3<x+2$ (since $x+2$ positive)

you can never have $x+3<x+2$.

$\underline{x<-2}$ need $-x-2>x+3>x+2$ (since $x+2$ negative)

$x+3$ always greater than $x+2$ and $-x-2>x+3$

means $\underline{x<-5/2}$

PTO

10. $|x^2-2|\le 1 \Rightarrow -1 \le x^2-2 \le 1$

i.e. $1 \le x^2 \le 3$

so either $\underline{1 \le x \le \sqrt{3}}$ or $\underline{-\sqrt{3} \le x \le -1}$.

11. $|x-1| > |1+x| + 1$

need to consider 3 cases: $x>1$, $-1<x<1$ and $x<-1$.

If $x>1$ then need $x-1 > x+1+1$ so $-1>2$ impossible.

If $-1<x<1$ then need $1-x > x+2$ so $x<-\frac{1}{2}$

If $x<-1$ then need $1-x > 1-1-x$ or $1>0$ always true.

Combine to get $\underline{x<-\frac{1}{2}}$.

12. $\dfrac{|x|-3}{-x^2-1} \geq 1$ means $3-|x| \geq x^2+1$

or $x^2+|x|-2 \le 0$

or $(|x|+2)(|x|-1) \le 0$

so require $-2 \le |x| \le 1$

i.e. $\underline{-1 \le x \le 1}$

NOTE in Q5, for example, it would be wrong to write $3 < x < -2$

this is a common mistake, but makes no sense since it suggests that $3 < -2$!