1. Inverse roots

Suppose that \( Q(x) = ax^2 + bx + c \) satisfies \( ac \neq 0 \) and has roots (i.e. solutions of \( Q(x) = 0 \)) \( \alpha \) and \( \beta \).

Show that the quadratic \( \tilde{Q}(x) = cx^2 + bx + a \) has roots \( \frac{1}{\alpha} \) and \( \frac{1}{\beta} \).

Extensions

(1) Show that if \( \alpha_1, \ldots, \alpha_n \) are the roots of the polynomial \( P \), where

\[
P(x) = a_0x^n + \ldots + a_{n-1}x + a_n \quad \text{with} \quad a_0a_n \neq 0,
\]

then the roots of \( \tilde{P} \) given by

\[
\tilde{P}(x) = a_nx^n + \ldots + a_0
\]

are \( \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \).

(2) Show that the roots of

\[
P_e(x) = a_0x^{2n} + a_1x^{2n-1} + \ldots + a_{n-1}x^{n+1} + a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0 \quad (a_0 \neq 0)
\]

are of the form \( \alpha_1, \ldots, \alpha_n, \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \).

(3) What can you say about the roots of

\[
P_+(x) = a_0x^{2n+1} + a_1x^{2n} + \ldots + a_{n-1}x^{n+1} + a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0
\]

and the roots of

\[
P_-(x) = a_0x^{2n+1} + a_1x^{2n} + \ldots + a_{n-1}x^{n+1} - a_nx^n - a_{n-1}x^{n-1} - \ldots - a_0?
\]

2. Trigonometric polynomials

The angle sum formula tells us that

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1.
\]

Find a similar expression involving powers of \( \cos \theta \) for \( \cos 3\theta \).

Extensions

(1) Find the roots of \( 4\sqrt{2}x^3 - 3\sqrt{2}x = 1 \).

(2) What is \( \cos \frac{\pi}{12} \)?