1. Inverse roots

Suppose that $Q(x) = ax^2 + bx + c$ satisfies $ac \neq 0$ and has roots (i.e. solutions of $Q(x) = 0$) α and β.

Show that the quadratic $\tilde{Q}(x) = cx^2 + bx + a$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Hint Let $x = \frac{1}{t}$ in the definition of $Q(x)$.

Answer Following the hint, $Q(x) = Q(\frac{1}{t}) = \frac{a}{t^2} + \frac{b}{t} + c = \frac{\tilde{Q}(t)}{t^2}$. Since $ac \neq 0$ neither of the roots of Q are zero so

$$\alpha^2 \tilde{Q}(\frac{1}{\alpha}) = Q(\alpha) = 0 \text{ which implies that } \tilde{Q}(\frac{1}{\alpha}) = 0.$$

Extensions

(1) Show that if $\alpha_1, \ldots, \alpha_n$ are the roots of the polynomial P, where

$$P(x) = a_0 x^n + \ldots + a_{n-1}x + a_n \text{ with } a_0 a_n \neq 0,$$

then the roots of $\tilde{P}$ given by

$$\tilde{P}(x) = a_n x^n + \ldots + a_0$$

are $\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$.

Answer Use the same trick as in the main question:

$$P(\frac{1}{t}) = \frac{a_n t^n + \ldots + a_0}{t^n} = \frac{\tilde{P}(t)}{t^n},$$

and, since none of the roots are zero,

$$\tilde{P}(\frac{1}{\alpha_i}) = 0.$$

(2) Show that the roots of

$$P_e(x) = a_0 x^{2n} + a_1 x^{2n-1} + \ldots + a_{n-1}x^{n+1} + a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 \ (a_0 \neq 0)$$

are of the form $\alpha_1, \ldots, \alpha_n, \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$.

Answer First, notice that $\tilde{P}_e = P_e$. It follows that

$$\{\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_{2n}}\} = \{\alpha_1, \ldots, \alpha_{2n}\}.$$

So every root is paired with its inverse (up to multiplicity, so if 2 is a double root then so is $\frac{1}{2}$).

(3) What can you say about the roots of

$$P_+(x) = a_0 x^{2n+1} + a_1 x^{2n} + \ldots + a_n x^{n+1} + a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

and the roots of

$$P_-(x) = a_0 x^{2n+1} + a_1 x^{2n} + \ldots + a_n x^{n+1} - a_n x^n - a_{n-1}x^{n-1} - \ldots - a_0?$$

Answer The same argument works in both cases (in the second case $\tilde{P}_e = -P_e$ but that doesn't affect the argument about the roots) but now there are $2n+1$ roots. This means that one root (at least) must be its own inverse. The two solutions of $x = \frac{1}{x}$ are 1 and -1. In the case of P_+ it's clear that -1 is a root. In the case of P_-, 1 is a root! The other roots in both cases will come in reciprocal pairs.

2. Trigonometric polynomials

The angle sum formula tells us that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1.$$

Find a similar expression involving powers of $\cos\theta$ for $\cos 3\theta$.

Hint Write $3x = 2x + x$!

Answer

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta)\\ &= \cos 2\theta\cos\theta - \sin 2\theta \sin\theta\\ &= 2\cos^3\theta - 2\cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - 2\cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

Extensions

(1) Find the roots of $4\sqrt{2}x^3 - 3\sqrt{2}x = 1$.

Hint Set $x = \cos\theta$. What is $\cos\frac{3\pi}{4}$?

Answer Following the first hint we get

$$4\sqrt{2}\cos^3\theta - 3\sqrt{2}\cos\theta = 1$$

or

$$\cos 3\theta = \frac{1}{\sqrt{2}}.$$

So

$$3\theta = \frac{\pi}{4} + 2n\pi,$$

and so

$$\theta = \frac{\pi}{12} + \frac{2n}{3}\pi.$$

The three corresponding values for $\cos\theta$ are $\cos\frac{\pi}{12}$, $\cos\frac{3\pi}{4}$ and $\cos\frac{17\pi}{12}$. The middle value is $-\frac{1}{\sqrt{2}}$ Now if we divide $4\sqrt{2}x^3 - 3\sqrt{2}x - 1$ by $x + \frac{1}{\sqrt{2}}$ we get the quadratic $Q(x) = 4\sqrt{2}x^2 - 4x - \sqrt{2}$. Then Q has roots $\frac{\sqrt{2}\pm\sqrt{6}}{4}$, so these are the other two roots of the equation.

(2) What is $\cos\frac{\pi}{12}$?

Answer Since $0 \le \frac{\pi}{12} < \frac{\pi}{2}$, $\cos\frac{\pi}{12}$ is positive, so it must be the positive root of Q, $\frac{\sqrt{2}+\sqrt{6}}{4}$.