1. Expanding Base I

Recall that the number of ways n distinct objects can be put in order is $n! = 1 \times 2 \times \dots n$.

Let

$$a_k = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-1}{k!}.$$

What is a_k as a simple fraction?

Hint Consider $b_k = a_k + \frac{1}{k!} = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-2}{(k-1)!} + \frac{k}{k!}$ and cancel the obvious common factor in the last fraction.

Extensions

(1) Find N_k, the number of sequences $x_2, \dots x_k$ with $x_2 = 0$ or 1, $x_3 = 0$ or 1 or 2, ..., $x_k = 0, 1, \dots$ or $k-1$.

Hint How many values can x_2 take? For each of these, how many values can x_3 take?

(2) Deduce that every fraction in the interval $[0, 1)$ of the form $x = \frac{m}{n!}$ (with m and n integers) can be written as

$$\frac{x_2}{2!} + \dots \frac{x_n}{n!}, \tag{1.1}$$

with $x_2 = 0$ or 1, $x_3 = 0$ or 1 or 2, ..., $x_n = 0, 1, \dots$ or $n-1$.

Hint Use (1).

(3) Remember that a number x is said to be *rational* if it can be written as a (possibly improper) fraction $\frac{m}{n}$ for some pair of integers m and n.

Show that every rational, x, in $[0, 1)$ can be written

$$\frac{x_2}{2!} + \dots + \frac{x_n}{n!},$$

with $x_2 = 0$ or 1, $x_3 = 0$ or 1 or 2, ..., $x_n = 0, 1, \dots$ or $n-1$ for some value of n.

Hint Use the previous part!