

Favourite Song

Question:

You have a playlist with exactly 100 tracks (i.e., songs), numbered 1 to 100. To go to another track, there are two buttons you can press:

- “Next” will take you to the next track in the list or back to song 1 if you are currently on track 100.
- “Random” will take you to a track chosen uniformly from among the 100 tracks. Pressing “Random” can restart the track you’re already listening to — this will happen 1 percent of the time you press the “Random” button.

For example, if you started on track 73, and you pressed the buttons in the sequence “Random, Next, Random, Random, Next, Next, Random, Next,” you might get the following sequence of track numbers: 73, 30, 31, 67, 12, 13, 14, 89, 90. You always know the number of the track you are currently listening to.

Your goal is to get to your favourite song (on track 42, of course) with as few button presses as possible.

What should your general strategy be? Assuming you start on a random track, what is the average number of button presses you would need to make to reach your favourite song?

Hints:

Think of a simple strategy first. How long does that take on average? Thinking of the cases that slow this down, can you come up with a more complex strategy?

Answer:

Clearly, it’s not a good idea to hope to land on Song 42 by chance. Speculating on landing a few tracks before Song 42 and got there using the Next-button also requires a lot of luck, and with bad luck one might end up just after it requiring to press the Next-button 99 times. On average, the combination of one random choice with Next-button with require to press it 49.5 times.

A more sensible strategy is to discard the worst results obtained with the Random-button. Instead, press as on the Random-button as long as it needs to get close enough, and then continue wit the Next-button until hitting 42. The question is now simply *what is close enough* to obtain, on average, the best result?

To find the optimal size of the Interval before 42, determine the formula for the average numbers of pressing the Next-button. The interval preceding 42 has the length n . I press the Random-button until I hit one of the positions between $42 - n$ and $42 - 1$. If I hit 42 I am already done, so that should also be included in the target interval.

With a total of 100 songs, the chance to land in the target interval is $(n + 1)/100$. Hence, I need to press, on average, $100/(n + 1)$ times to get there. Since the first move doesn't count I can subtract 1, so the number is $100/(n + 1) - 1$. To get to 42 from there, on average, I need to press the Next-button

$$(n + (n - 1) + (n - 2) + \dots + 1 + 0)/(n + 1)$$

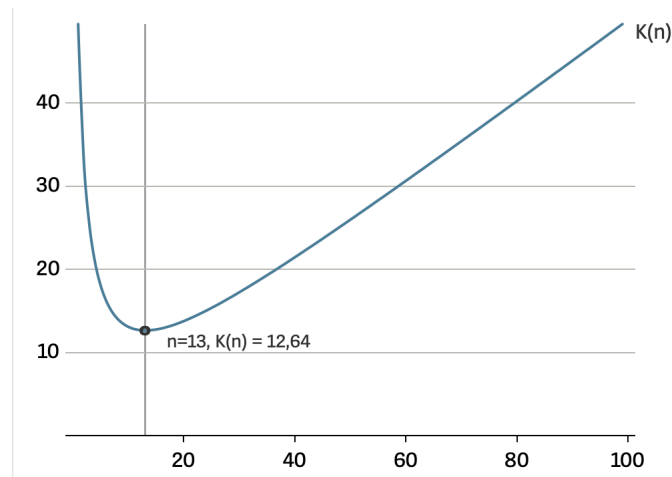
times. This equals

$$(n(n + 1) + n(n + 1)/2)/(n + 1) = n - n/2 = n/2.$$

So the average total number of I have a press buttons is

$$K(n) = 100/(n + 1) + n/2 - 1.$$

For which n is $K(n)$ the smallest? We can find this by plotting the function and then taking the nearest larger integer. This is 13.



Hence, the strategy is to press on the Random-button until we hit one of the songs between track number 29 and 42. If we are in 42 we are done. Otherwise we press the Next-button until we are there which, on average, is 12.64 times.