Alice and Bob

You play a game with Alice and Bob:

1. They pick one number each, with the only constraint that \( A \neq B \), meaning that they are not allowed to pick the same number. (However, they can use any strategy: communicate with each other, use randomness to pick their numbers, etc.)

2. You toss a coin to choose who reveals their number, Alice or Bob.

3. After seeing the revealed number, you are to guess who has the bigger number.

Find a strategy so that, if playing this game repeatedly, you win more often than you lose.

**Hint**

Assume first that they pick numbers from the faces of a die, that is, from the set \( S = \{1, 2, 3, 4, 5, 6\} \) and that your strategy is deterministic. For example, if the revealed number was > 3, you always declared that it was the bigger of the two numbers. Would your strategy always work? What if Alice and Bob always picked 4 and 5? What if \( S \) is a different set, e.g. the real numbers? Can you think of a more abstract description of the strategy that enables you to introduce a dependency on what you see?

**Answer**

For your strategy you first pick a function \( \psi \) on \( S \) that is strictly increasing, that goes to 0 for \( X \) to \( -\infty \) and that goes to 1 for \( X \) to \( \infty \).

Then you declare the revealed number \( X \) (where \( X \) may be \( A \) or \( B \)) bigger with probability \( \psi(A) \), where \( \psi \) is a cumulative distribution function. (It can be any such function.

Why it works:
Without loss of generality, assume \( A < B \). After tossing the coin, the chance that you see \( A \) or \( B \) is the same and equals 1/2. However, what is the chance your answer is correct?

\[
P(\text{win}) = P(\text{seen } B)P(\text{declared } B \text{ as bigger given } B \text{ was seen })
+ P(\text{seen } A)P(\text{declared } B \text{ as bigger given } A \text{ was seen })
\]

\[
= \frac{1}{2}\psi(B) + \frac{1}{2}(1 - \psi(A))
= \frac{1}{2} + \frac{1}{2}(\psi(B) - \psi(A))
> \frac{1}{2},
\]

because \( \psi \) is strictly monotone and \( A < B \).
Notes

- An interpretation of this is that it is a probabilistic strategy and $\psi$ is a cumulative distribution function. An examples for a cumulative functions if $S$ is the real numbers is the normal distribution. In the case where $S$ is a discrete set, the function $\psi$ does not have to be strictly increasing, but could be piecewise constant and jump up by a value larger than 0 in each values of $S$, e.g. $\psi(k) = k/6$ ($k = 1, 2, \ldots, 6$) for the die. (For more details about such functions see [https://en.wikipedia.org/wiki/Cumulative_distribution_function](https://en.wikipedia.org/wiki/Cumulative_distribution_function)).

- The strategy always works no matter whether the set $S$ is finite or infinite, discrete or continuous.

- What does "without loss of generality" means? It means that the argument works in the same way without that assumption. Here, the assumption was $A < B$. The case of equality was excluded. If $A > B$ then the argument can just be rewritten with the roles of $A$ and $B$ exchanged.