Triangle mosaic

Question:

Select a starting point P_0 and draw a 1cm long line c_1 ending in P_1.

From P_1, make a 1cm line perpendicular to $\overline{P_0P_1}$. Connect its other end point P_2 with P_0 to obtain a triangle and call the hypotenuse c_2.

From P_2, make a 1cm line perpendicular to $\overline{P_0P_2}$ (away from the triangle). Connect its other end point P_3 with P_0 to obtain a triangle and call the hypotenuse c_3.

Keep going. Step k looks like this:

From P_{k-1}, make a 1cm line perpendicular to $\overline{P_0P_{k-1}}$ (away from the previous triangle). Connect its other end point P_k with P_0 to obtain a triangle and call the hypotenuse c_k.

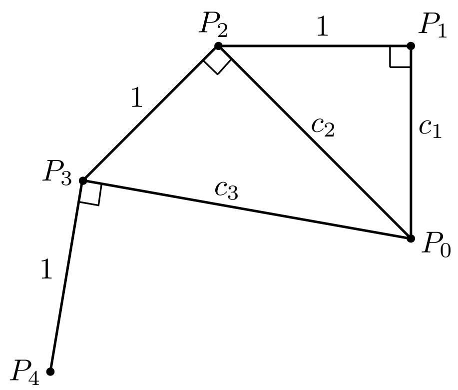
Denote the angles between c_k and c_{k+1} with α_k for $k = 1, 2, 3, \ldots$

Let c_n be the first line to be more than one complete turn away from the starting line c_1.

What is n? Derive formulas for c_k and α_k for $k = 1, 2, 3, \ldots$

Hints:

Construct the first few triangles on pencil and paper. Here is a start:



To find a formula for c_k you could derive this for $k = 1, 2$, and 3, guess it for a general k, and then prove your conjecture using the technique of *induction*. If you have not learned this or you can not remember it, here are some resources:

- Section 2 in a proof technique handout from Dartmouth: `https://math.dartmouth.edu/~m22x17/misc/LaLonde2012_proof_techniques.pdf`
- Video tutorial by Kimberly Brehm: `https://www.youtube.com/watch?v=TqpNDiqsz7k`
- Guidance for Year 11 and Year 12 (Australian) teachers with many examples including *Tower of Hanoi* and a two-colour problem: `https://www.amsi.org.au/teacher_modules/pdfs/Maths_delivers/Induction5.pdf`

To find a formula for α_k use trigonometry and the formula for c_k.

Solution:

Using Pythagoras' theorem, $c_2^2 = c_1^2 + 1^2 = 2$, so $c_2 = \sqrt{2}$. By the same token, $c_3^2 = c_2^2 + 1^2 = 3$, so $c_3 = \sqrt{3}$, $c_4 = c_3^2 + 1^2 = 3 + 1 = 4$, so $c^4 = \sqrt{4}$, and so on, leading to the claim that for any $k = 1, 2, 3, \ldots,$

$$\text{(Claim(k))} \qquad c_k = \sqrt{k}$$

To prove that formally, we use the technique of *induction.* The *base case* $k = 1$ is true by construction: $c_1 = 1 = \sqrt{1}$. For the *induction step,* we need to show that, for any $k = 1, 2, 3, \ldots,$ if $((I(k)))$ is true then $(I(k+1))$ is also true.

Assume that (Claim(k)) is true and show that

$$\text{(Claim(k+1))} \qquad c_{k+1} = \sqrt{k+1}.$$

To do that, we proceed just as in the examples above, but for the general case. Using first Pythagoras' theorem and then (Claim(k)), $c_{k+1}^2 = c_k^2 + 1^2 = \sqrt{k}^2 + 1 = k + 1$, which implies (Claim(k+1)).

To obtain the angles note that for any $k = 1, 2, 3, \ldots$, $\sin \alpha_k = 1/c_{k+1}$, so $\alpha_k = \arcsin 1/c_{k+1} = 1/\sqrt{k}$.

To find the first line c_n that is more than one turn away from c_1, find the smallest n such that

$$\sum_{k=1}^{n-1} \alpha_k > 2\pi.$$

(That means, keep adding α_k until their sum is larger than 2π.)

At this stage you may use a calculator to find the answer. You can also write a short programme in R, Python or some other programming language. See below for sample code in R to do that.

You will find it needs 17 triangles. Te first line to be more than one complete round away from the starting line is therefore c_{18}, so $n = 18$.

If your $18th$ birthday is around this time: **Happy birthday to you!**

Sample R code:

```r
c <- c()
alpha <- c()
n = 1
c[1] = 1
repeat{
        c[n+1] <- n+1
        alpha[n] <- asin(1/sqrt(c[n+1]))
        n <- n+1
        print(" ")
        if ( sum(alpha) >= 2*pi ){break}
}
```