

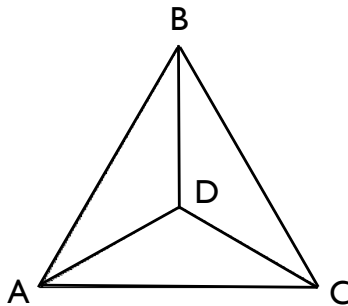
### 1998 STEP III, q12 solution

The villages  $A, B, C, D$  lie at the vertices of a tetrahedron, with each pair joined by a road. After a snowfall the probability any road is blocked is  $p$  and is independent of the conditions of any other road, Show that the probability that it is possible, after a snowfall, to travel from any village to any other village by some route is

$$P = 1 - p^3(6p^3 - 12p^2 + 3p + 4).$$

Do we understand the question: what are the technical words? Tetrahedron and Independent.

To begin with, it makes sense to sketch the picture:



(Here we draw a birds-eye view of the mountain, so that roads do not overlap on the sketch.)

Are there any special cases when we can determine if the travel is possible or not between all villages?

Clearly, if there are no roads blocked, then we can travel from any village to any other. On the other hand, if all the roads are blocked, then there are no pairs of villages we can travel between.

*This suggests a plan:* it might be helpful to start breaking the possible outcomes down into the events

$$E_k := \{k \text{ roads blocked}\}, \quad k = 0, 1, \dots, 6,$$

and considering what happens in each of these disjoint cases (dealing with the easier cases first!):

- We have already noted that, if  $E_0$  occurs, then all villages are connected.
- Similarly, if  $E_6$  occurs, then no villages are connected.
- If  $E_1$  occurs, so only 1 road is blocked, and you can still travel from any village to any other village by taking an alternative route around the blocked road.
- Also, if  $E_5$  occurs, then only two villages are connected, and so it is not possible to travel from any village to any other village.
- Now, suppose  $E_2$  occurs. *Now its very helpful to use the symmetry of the tetrahedron!* Suppose that  $AB$  is one of the roads blocked. There are two other possibilities we have to consider.

First:  $AC$  is blocked. In this case, it is possible to travel from any village to any other by passing through  $D$ .

Second:  $CD$  is blocked. In this case, we have an unblocked loop that passes through all villages. In particular, in either case, we can travel from any village to any other village.

Using the symmetry, any other configuration in which two roads are blocked, is equivalent (if we relabelled the vertices) to one of these two cases. So we conclude if  $E_2$  occurs, then travel is always possible.

- Next, suppose  $E_4$  occurs. In this case we have 2 unblocked roads. There are two cases.

First: If these meet at a village, then these only connect 3 of the four villages.

Alternatively: If they do not, then the network is broken into two disconnected components.

Either way, we can not travel from any village to any other village.

- Finally, if  $E_3$  occurs, then 3 roads are blocked. Since these 3 roads have 6 ends and there are only 4 vertices, it must be the case that at least two of them meet at a village. Suppose these roads are  $AB$  and  $AC$ . There are two possibilities to consider:

First: It is easy to see from the sketch that if any road from the triangle  $BCD$  is the third blocked road, then the network is still connected.

Alternatively: On the other hand, if  $AD$  is the third blocked road, then  $A$  is separated from the other villages.

Using symmetry, it follows that if  $E_3$  occurs, then the only way in which the network can become disconnected is if each of the blocked roads is adjacent to a single village.

Now, we note that the expression for  $P$  given in the question starts with a “1–”. Since

$$\begin{aligned} P &= \mathbf{P}(\text{we can travel from any village to any other village}) \\ &= 1 - \mathbf{P}(\text{we can not travel from any village to any other village}), \end{aligned}$$

this suggests it will be best to try to compute the probability that we can not travel from any village to any other village. Breaking this down using the disjointness of the events  $E_k$ ,  $k = 0, 1, \dots, 6$ , and using our discussion of what happens in the various cases above, this is equal to

$$\mathbf{P}(E_6) + \mathbf{P}(E_5) + \mathbf{P}(E_4) + \mathbf{P}(E_3 \text{ and the blocked roads are all adjacent to a single village})$$

Now to calculate these probabilities we use independence and count configurations. Multiplying together the probabilities that each road is blocked gives

$$\mathbf{P}(E_6) = p^6$$

When  $E_5$  occurs there are six possibilities for the open road giving

$$\mathbf{P}(E_5) = 6(1-p)p^5$$

When  $E_4$  occurs there are  $15 = \binom{6}{2}$  possibilities for the open road giving

$$\mathbf{P}(E_5) = 15(1-p)^2p^4$$

*Reflect: these are the probabilities from the Binomial distribution* Finally, we recall that if  $E_3$  occurs, then the only way in which the network can become disconnected is if each of the blocked roads is adjacent to a single village. Since there are 4 vertices, there are 4 ways in which this can happen. So,

$$\mathbf{P}(E_3 \text{ occurs and we can not travel from any village to any other village}) = 4p^3(1-p)^3.$$

Putting all the pieces together, we thus obtain

$$\begin{aligned} P &= 1 - 4p^3(1 - 3p + 3p^2 - p^3) - 15p^4(1 - 2p + p^2) - 6p^5(1 - p) - p^6 \\ &= 1 - p^3(6p^3 - 12p^2 + 3p + 4). \end{aligned}$$

(Note this differs from the equation given in the actual exam paper, which contained an unfortunate error!)