Estimating functions and composite likelihood for Cox point processes.

Gunnar Hellmund, Thiele Centre, University of Aarhus

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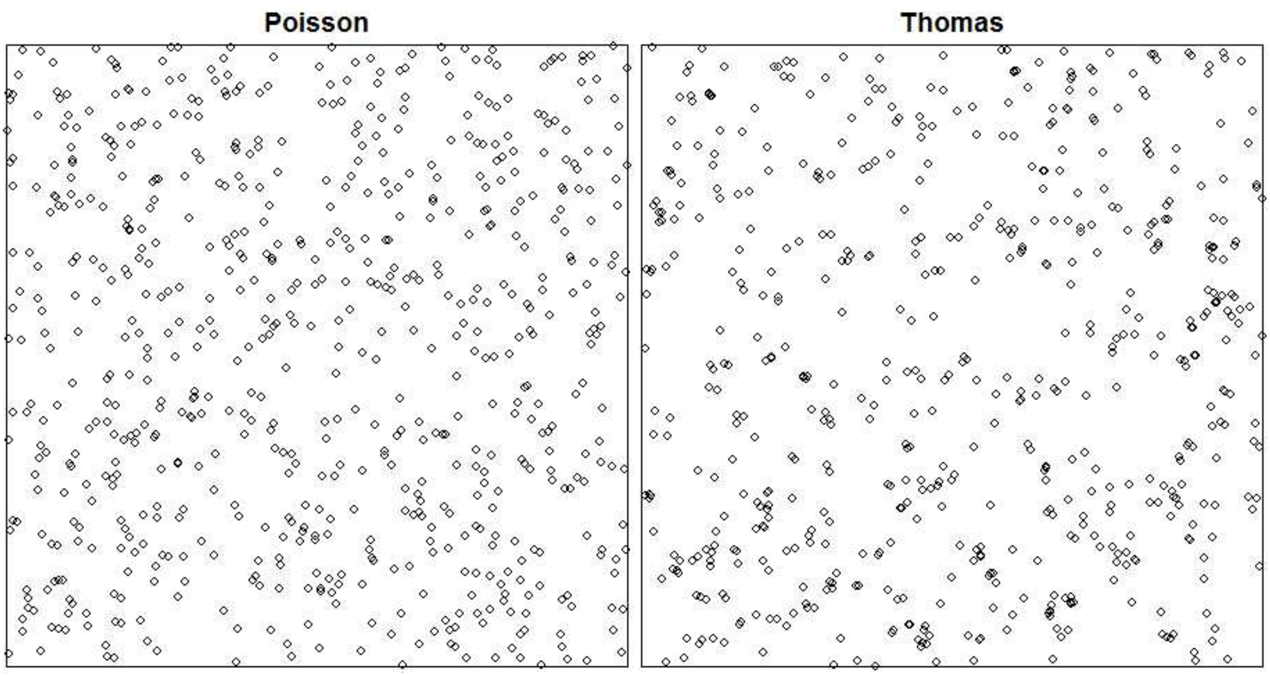
Cox point processes

A Cox process $\mathbf{X}$ on a subset W of $\mathbb{R}^d$ for some $d \geq 1$ is defined in terms of a random intensity function Λ, such that given $\Lambda = \lambda$, $\mathbf{X}$ is a Poisson process with intensity λ.

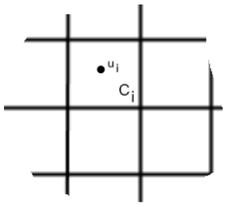


Cox point processes

For $k \geq 1$ the k'th order mean intensity function is defined as

$$m^k\left(\mathbf{x}_1, \ldots, \mathbf{x}_k | \theta\right) = E_\theta\left[\Lambda\left(\mathbf{x}_1\right) \cdot \cdots \cdot \Lambda\left(\mathbf{x}_k\right)\right].$$

Partition:
$W = \cup_i C_i$ where C_i small cells with volume $|C_i|$ and $u_i \in C_i$.



Setting $N_{i_1,\ldots,i_k} = 1\left[\#\left(\mathbf{X} \cap C_{i_j}\right) > 0, j = 1, \ldots, k\right]$, then

$$P_\theta\left(N_{i_1,\ldots,i_k} = 1\right) \approx m^k\left(u_{i_1}, \ldots, u_{i_k}; \theta\right)\left|C_{i_1} \times \cdots \times C_{i_k}\right|.$$

Composite Likelihood

Guan suggested composite likelihood

$$\begin{aligned} L_G(\theta) &= \sum_{\mathbf{x},\mathbf{y},|\mathbf{x}-\mathbf{y}|<d}^{\neq} \ln m^2\left(\mathbf{x}, \mathbf{y}|\theta\right) \\ &\quad -\#\left(|\mathbf{x}-\mathbf{y}| < d\right) \ln \int_{|\mathbf{x}-\mathbf{y}|<d} m^2\left(\mathbf{x}, \mathbf{y}|\theta\right) d\left(\mathbf{x}, \mathbf{y}\right). \end{aligned}$$

$\#\left(|\mathbf{x}-\mathbf{y}| < d\right)$ number of pairs of points with distance less than d.
Was only justified for stationary Cox pp.
$\nabla L_G(\theta)$ unbiased.

Composite Likelihood

A Bernoulli composite likelihood can be formed

$$L_B(\theta) = \sum_{\mathbf{x},\mathbf{y},|\mathbf{x}-\mathbf{y}|<d}^{\neq} \ln m^2(\mathbf{x},\mathbf{y}|\theta) - \int_{|\mathbf{x}-\mathbf{y}|<d} m^2(\mathbf{x},\mathbf{y}|\theta)\, d(\mathbf{x},\mathbf{y}).$$

$\nabla L_B(\theta)$ is unbiased.
Numerically the Bernoulli composite likelihood seems to be more well-behaved.

Bernoulli composite likelihood

The Bernoulli composite likelihood based on $N_{i_1,\ldots,i_k} = 1\left[\#\left(\mathbf{X}\cap C_{i_j}\right) > 0, j = 1,\ldots,k\right]$ (and partition $\mathbf{C}$) is given as

$$\begin{aligned} & \prod_{\mathbf{i}\in I^k} p_{i_1,\ldots,i_k}(\theta)^{N_{i_1,\ldots,i_k}} \left(1 - p_{i_1,\ldots,i_k}(\theta)\right)^{1-N_{i_1,\ldots,i_k}} \\ \approx & \prod_{\mathbf{i}\in I^k} m^k\left(u_{i_1},\ldots,u_{i_k};\theta\right)^{N_{i_1,\ldots,i_k}} \left|C_{i_1}\times\cdots\times C_{i_k}\right|^{N_{i_1,\ldots,i_k}} \\ & \cdot\left(1 - m^k\left(u_{i_1},\ldots,u_{i_k};\theta\right)\left|C_{i_1}\times\cdots\times C_{i_k}\right|\right)^{1-N_{i_1,\ldots,i_k}} \end{aligned}$$

where $p_{i_1,\ldots,i_k}(\theta) = P_\theta\left(N_{i_1,\ldots,i_k} = 1\right)$.

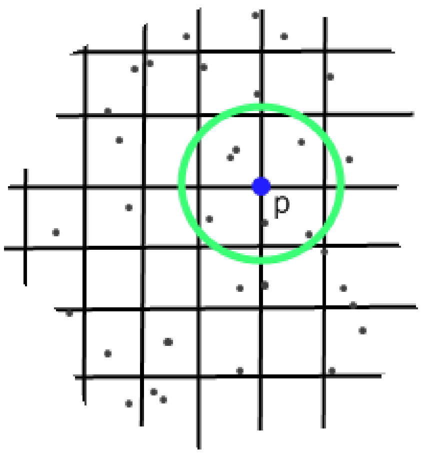
Bernoulli composite likelihood

Under suitable regularity conditions when the cell sizes tend to zero the log composite likelihood becomes

$$\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_k \in \mathbf{X}}^{\neq} \ln\left(m^k(\mathbf{x}_1,\ldots,\mathbf{x}_k;\theta)\right) - \int_{W^k} m^k(\mathbf{x};\theta)\, d\mathbf{x}$$

General setup

Point process $\mathbf{X}$ defined on $\mathbb{R}^d$.
Put a grid on top of $\mathbb{R}^d$. For each point p in the grid, define unbiased estimating function $\nabla \ell_p$, depending on $\mathbf{X}$ restricted to the surroundings of p.



Theorems for consistency and a CLT for the zeros of $\frac{1}{n}\sum_{i=1}^{n} \nabla \ell_{p_i}$.
Crowder (consistency for estimating functions),
Gyuon (CLT for mixing non-stationary random fields on $\mathbb{Z}^d$)

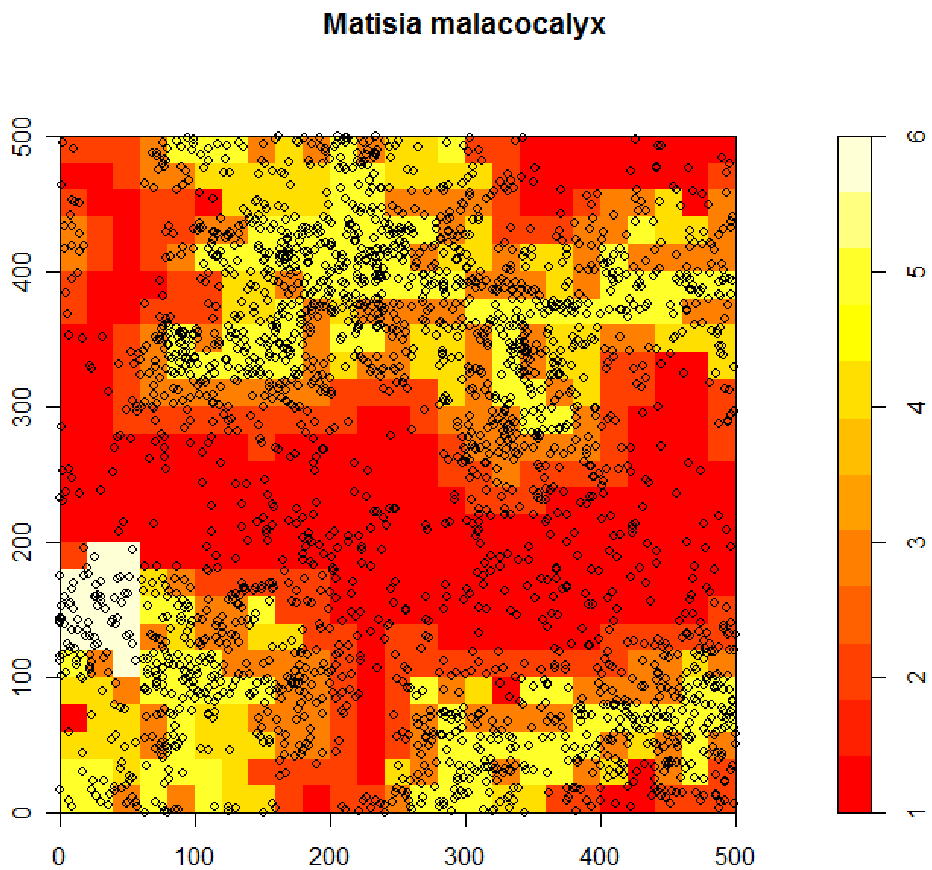
Example

Inhomogeneous Thomas process is a cluster process, with random intensity function:

$$\Lambda(\mathbf{x}) = \sum_{\mathbf{c}\in\Phi} e^{c_1+\beta\cdot z(\mathbf{x})} \frac{1}{2\pi\omega^2} e^{-\frac{|\mathbf{x}-\mathbf{c}|^2}{2\omega^2}}$$

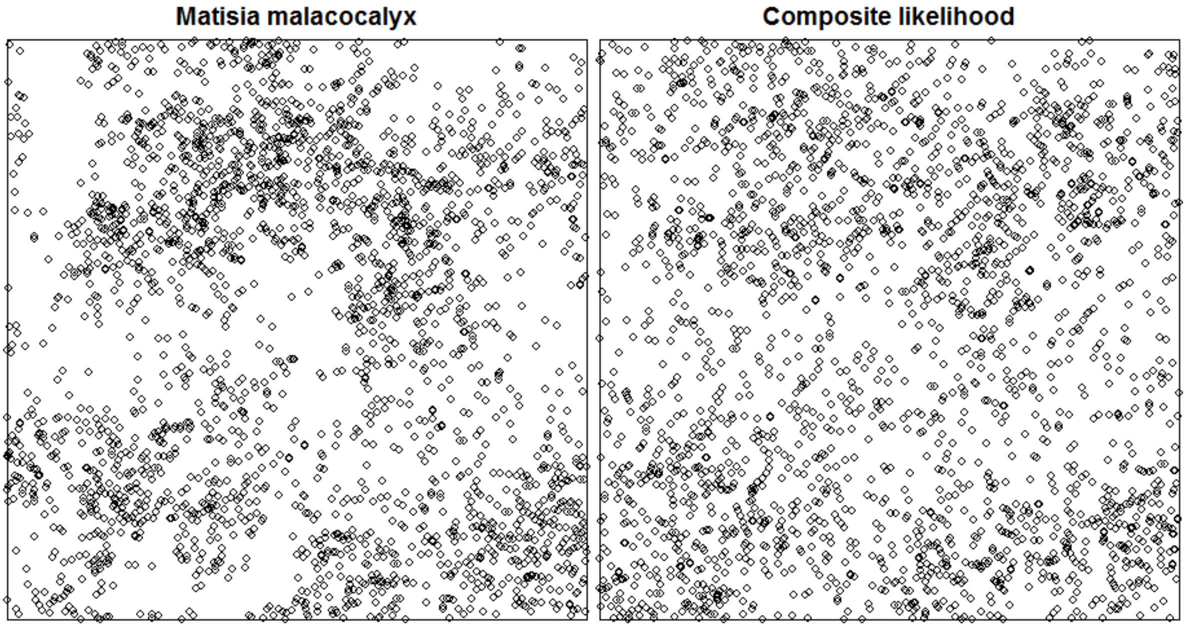
c_1 scaling parameter,β describes the effect of the co-variates, z is the co-variate information vector function and Φ has intensity κ.

Example



Example

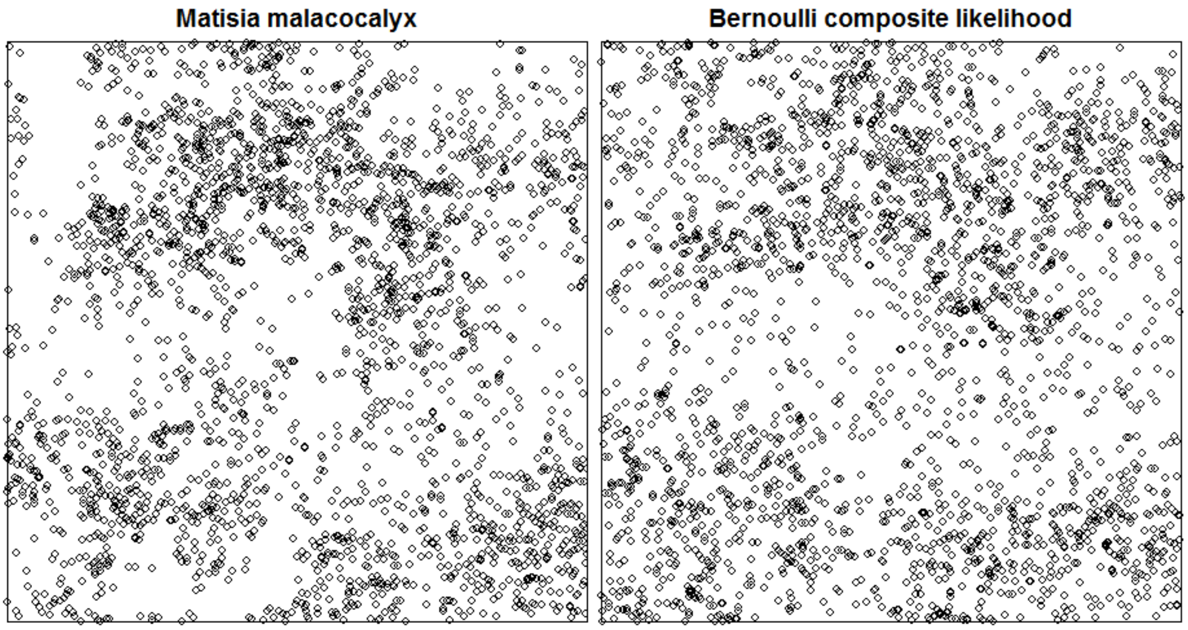
First type of composite likelihood (based on article by Guan).
Simulation from estimated distribution



Estimates:
κ=exp (-2.01),c_1=-2.49, β=(-0.81;-0.37;0.18;-0.14;0.29),
ω^2=11.89

Example

Bernoulli composite likelihood.
Simulation from estimated distribution



Estimates:
κ=exp (-2.11),c_1=-2.38, β=(-0.97;-0.48;0.24;-0.21;0.44),
ω^2=11.61

Simulation

A typical parametric bootstrap result concerning the variance for the Gaussian kernel (for inhom. Thomas pp. model with $\ln \omega^2 = 4$)
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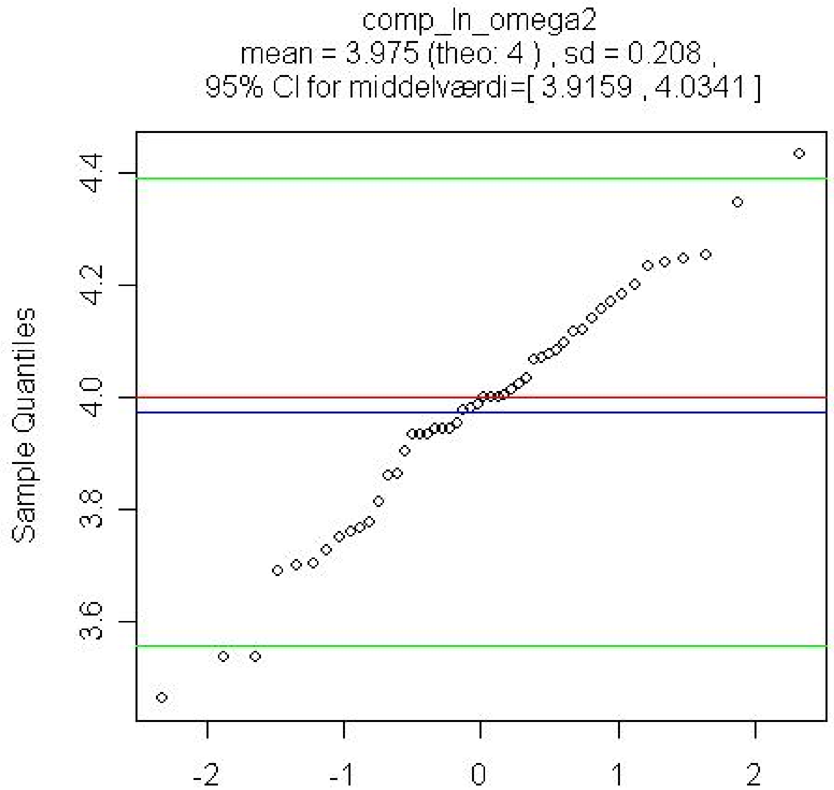

Moment estimation

Using the general framework (for consistency and CLT) we have furthermore studied a moment based estimation function:

$$L_M(\theta) = -\sum_i \sum_{j=1}^{k} a_{i,j} \left(\# (\mathbf{X} \cap C_i)^j - E_\theta \left[\# (\mathbf{X} \cap C_i)^j \right] \right)^2$$

where $a_{i,j}$ are positive constants.
Estimates behave very nice in simulation studies - less nicer applied to rain-forest data.
Method is extremely fast - and can be used to generate startvalues for composite likelihood estimation.

Implementation details

All methods were implemented in Fortran 95/2003
Computions was performed using a PC with a Q6600 Intel processor, 2GB RAM, 10krpm SATA HD.

Speed (for the given example):

Composite likelihood methods $\sim$ 30sec.

Moment estimation < 1 sec.

Future investigations

1. Estimation in several classes of Cox point processes (rain forest data). Using combinations of log-Gaussian and Cox cluster processes (with different types of kernels)
2. Bootstrap-methods
3. Methods for selection of appropriate Cox point processes.

Thank You for your attention

Gunnar Hellmund

hellmund@imf.au.dk

THIELE CENTRE

joint work with

Ph.D. Rasmus Plenge Waagepetersen,

Spar Nord/University of Aalborg