Criteria for subgeometric ergodicity of strong Markov processes

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Let

- $\{\Phi_t\}_{t \geq 0}$ be a time-continuous strong Markov process on X ($\mathsf{X} = \mathbb{R}^d$), with cadlag paths.
- $\{P^t\}_{t \geq 0}$ be the Markov semigroup : $\mathbb{P}_x(\Phi_t \in A) = P^t(x, A)$.

Objective

Conditions that imply

$$\lim_{t\to\infty} r(t) \sup_{|g|\le f} |P^t g(x) - \pi(g)| = 0 \qquad \forall x \in \mathsf{X}, \qquad (1)$$

π : (the) invariant probability measure

f : positive function,

Ex. $f = 1$ [Total variation norm], $f(x) \sim |x|^p$, $\cdots$.

r : positive non-decreasing rate function

Subgeometric rate function:

Ex. $r(t) \sim t^\tau \quad r(t) \sim \{\log t\}^\tau \quad r(t) \sim \exp(ct^\gamma)\ {\scriptstyle 0<\gamma<1}$.

$$0 < \liminf_t r(t)/\tilde{r}(t) \le \limsup_t r(t)/\tilde{r}(t) < \infty, \qquad \lim_t \frac{\log \tilde{r}(t)}{t} \downarrow 0.$$

▶ Answer : (1) $\Leftarrow$ Return-time to a petite set $\Leftarrow$ Drift inequalities

$\Downarrow$

Regular set and regular measures

Skeleton chain, resolvent

Moderate deviation principle

Definitions

Delayed return-time to A:

$$\tau_A(\delta) = \inf\{t \geq \delta, \Phi_t \in A\}, \qquad \tau_A(0) = \tau_A.$$

Phi-irreducibility: there exists a measure ϕ

$$\phi(A) > 0 \implies \forall x, \qquad \mathbb{P}_x(\tau_A < \infty) > 0$$

Maximal irred. measure: irred. measure ψ s.t. $\phi \prec \psi$.
Accessible set: a set s.t. $\psi(A) > 0$.

Petite set C: there exist a probability measure a on $[0, \infty)$ and ν_a measure on X

$$\int_0^\infty P^t(x, \cdot) \; a(dt) \geq \nu_a(\cdot), \qquad \forall x \in C.$$

Existence: A phi-irreducible process possesses an accessible closed petite set.

Aperiodicity: there exist an accessible petite set C and $t_0 > 0$ such that

$$\inf_{x \in C} P^t(x, C) > 0, \qquad \forall t \geq t_0.$$

Discrete-time / Continuous-time

There exist a rate function r and a function f such that

$$\lim_{t\to\infty} r(t) \sup_{|g|\leq f} |P^t g(x) - \pi(g)| = 0 \qquad \forall x \in \mathsf{X},$$

Discrete-time case :

- phi-irreducible, aperiodic,
- C petite set such that $\sup_{x\in C} \mathbb{E}_x \left[\sum_{k=0}^{\tau_C - 1} \tilde{r}(k) f(\Phi_k)\right] < \infty$,

when
- $r(t) = \tilde{r}(t) = 1$,
- $r(t) \sim \kappa^t$, $\kappa > 1$, $r \neq \tilde{r}$ Meyn & Tweedie, 1993
- $r = \tilde{r}$ subgeom. Jarner Roberts 2002, Fort Moulines 2003, Douc et al. 2004

Continuous-time case :

- phi-irreducible, aperiodic,
- C closed petite set s.t. $\sup_{x\in C} \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \tilde{r}(s) f(\Phi_s) ds\right] < \infty$,

when
- $r(t) = \tilde{r}(t) = 1$ Meyn & Tweedie 1993
- $r(t) \sim \kappa^t$, $\kappa > 1$ $r \neq \tilde{r}$ Meyn Tweedie 1993, Down et al. 1995
- $\hookrightarrow$ subgeometric rate function ?

Res. 1

Theorem (subgeometric f-ergodicity) Fort & Roberts, 2005

If
- The process is Harris-recurrent with invariant prob. distribution π.
- Some skeleton chain P^m is phi-irreducible.
- There exist a closed petite set C, $\delta > 0$, a function $f_* \geq 1$ and a subgeometric rate function r_* s.t.

$$\sup_{x \in C} \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} r_*(s) ds \right] < \infty, \qquad \sup_{x \in C} \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} f_*(\Phi_s) ds \right] < \infty.$$

- There exists c such that $\sup_{t \leq m} P^t f_* \leq c f_*$.

Then
- $\pi(f_*) < \infty$.
- $\lim_{t \to \infty} r(t) \ \sup_{|g| \leq f} \ |P^t g(x) - \pi(g)| = 0, \qquad x \in \mathcal{S}_*$,

$$\begin{array}{lll} r(t) = r_*(t) & r(t) = 1 & r(t) = \Psi_1(r_*(t)) \\ f(t) = 1 & f(t) = f_*(t) & f(t) = \Psi_2(f_*(t)) \end{array}, \quad \Psi_1(x)\Psi_2(y) \leq x + y$$

- and similar conclusions

$$U_{r_*}(x) = \mathbb{E}_x\left[\int_0^{\tau_C(\delta)} r_*(s)ds\right], \qquad U_{f_*}(x) = \mathbb{E}_x\left[\int_0^{\tau_C(\delta)} f_*(\Phi_s)ds\right].$$

- If μ is a probability measure such that $\mu(U_{r_*} + U_{f_*}) < \infty$,

$$\lim_{t\to\infty} r(t) \sup_{|g|\leq f} |\int \mu(dx)P^t g(x) - \pi(g)| = 0.$$

- For all $x \in \mathsf{X}$,

$$r(t) \sup_{|g|\leq f} |P^t g(x) - \pi(g)| \leq c\{U_{r_*}(x) + U_{f_*}(x)\}.$$

- For all $x, y \in \mathsf{X}$,

$$\int_0^\infty r(t) \sup_{|g|\leq f} |P^t g(x) - P^t g(y)| dt \leq c\{U_{r_*}(x) + U_{f_*}(x) + U_{r_*}(y) + U_{f_*}(y)\}.$$

- If ∂r is a subgeometric rate function,

$$\int_0^\infty \partial r(t) \sup_{|g|\leq f} |P^t g(x) - \pi(g)| dt \leq C\{U_{r_*}(x) + U_{f_*}(x)\}.$$

Res. 2 : Sufficient Conditions for delayed return-time

Find $\quad \mathbb{E}_x\left[\int_0^{\tau_C(\delta)} r_*(s)ds\right] \leq V_{r_*}(x), \quad \mathbb{E}_x\left[\int_0^{\tau_C(\delta)} f_*(\Phi_s)ds\right] \leq V_{f_*}(x).$

Theorem (Answer 1) Douc, Fort & Guillin, 2006

If For all $x \in \mathsf{X}$,

$s \mapsto V(\Phi_s) - V(X_0) - \int_0^s \{-\phi \circ V(\Phi_u) + b\mathbb{1}_C(u)\}du$ is a $\mathbb{P}_x$-supermartingale

C closed set, $V \geq 1$ cadlag, $\phi > 0 \uparrow$ differentiable concave function.

Then for all $x \in \mathsf{X}$, $\delta > 0$,

$$\mathbb{E}_x\left[\int_0^{\tau_C(\delta)} r_\phi(s)ds\right] \leq \tilde{c}_\delta \; V(x), \quad \mathbb{E}_x\left[\int_0^{\tau_C(\delta)} \phi \circ V(\Phi_s)ds\right] \leq c_\delta \; V(x),$$

where $\qquad r_\phi(s) = \phi \circ H_\phi^{-1}(s), \qquad H_\phi(s) = \int_1^s \frac{du}{\phi(u)}.$

Rmk : r_ϕ subgeometric rate function if $\lim_{t\to\infty} \phi'(t) = 0$.

Res. 2 : Sufficient conditions for delayed return-time

Let $\mathcal{A}$ be the extended generator with domain $\mathcal{D}(\mathcal{A})$: $V \in \mathcal{D}(\mathcal{A})$ iff

$$s \mapsto V(\Phi_s) - V(\Phi_0) - \int_0^s \mathcal{A}V(\Phi_u)du \qquad \mathbb{P}_x\text{-local martingale.}$$

Theorem (Answer 2) Fort & Roberts 2005; Douc, Fort & Guillin, 2006

If for all $x \in \mathsf{X}$,

$$\mathcal{A}V(x) \leq -\phi \circ V(x) + b\mathbb{1}_C(x),$$

C closed set, $V \geq 1$ cadlag in $\mathcal{D}(\mathcal{A})$, $\phi > 0$ increasing differentiable concave function.

Then the previous theorem applies :

$$r_*(s) = r_\phi(s), \qquad f_*(x) = \phi \circ V(x).$$

Example Douc et al. 2004

- $\phi(t) \sim t^{1-\alpha} \longrightarrow r_\phi(t) \sim t^{1/\alpha - 1}$ $\quad 0 < \alpha < 1$.
- $\phi(t) \sim (1 + \log t)^\alpha \longrightarrow r_\phi(t) \sim \log^\alpha(t)$ $\quad \alpha > 0$.
- $\phi(t) \sim t/(\log t)^\alpha \longrightarrow r_\phi(t) \sim t^{-\alpha/(1+\alpha)} \exp(ct^{1/(1+\alpha)})$ $\quad \alpha > 0$.

Drift inequalities : continuous-time / discrete-time

	Continuous-time case	Discrete-time case
Non explosivity	$\mathcal{A}V \leq cV$	
Recurrence	$\mathcal{A}V \leq c\mathbb{1}_C$	$PV - V \leq b\mathbb{1}_C$
f-ergodicity	$\mathcal{A}V \leq -f + b\mathbb{1}_C$	$PV - V \leq -f + b\mathbb{1}_C$
Geometric ergodicity	$\mathcal{A}V \leq -cV + b\mathbb{1}_C$	$PV - V \leq -cV + b\mathbb{1}_C$ $\quad 0 < c < 1$
Polynomial ergodicity	$\mathcal{A}V \leq -cV^{1-\alpha} + b\mathbb{1}_C$	$PV - V \leq -cV^{1-\alpha} + b\mathbb{1}_C$
Subgeometric ergodicity	$\mathcal{A}V \leq -\phi \circ V + b\mathbb{1}_C$	$PV - V \leq -\phi \circ V + b\mathbb{1}_C$

Example: Elliptic diffusions on $\mathbb{R}^n$ (a)

Consider

$$\Phi_t = \Phi_0 + \int_0^t b(\Phi_s)ds + \int_0^t \sigma(\Phi_s)dB_s,$$

Conditions on b, σ

- σ bounded; b, σ locally lipschitz.
- $a(x)$ non singular.
- There exist $0 < p < 1$, $M, r > 0$ s.t.

$$\langle b(x), x \rangle \leq -r|x|^{1-p} \qquad |x| \geq M.$$

Drift inequalities

Itô's formula yields

$$\mathcal{A}V(x) = \langle b(x), \nabla V(x) \rangle + \frac{1}{2}\mathrm{Tr}\left(\nabla^2 V(x)\ a(x)\right),$$

Ex. $V(x) = \exp(\iota |x|^m)$.

Example: Elliptic diffusions on $\mathbb{R}^n$ (b)

Set $$\lambda_+ := \sup_{x\neq 0} |x|^2 \langle a(x)x, x\rangle.$$

Theorem (Ergodicity of the diffusion)

π-integrable $\exists\pi$ and for all $c > 0$ s.t. $r - 0.5c\lambda_+(1-p) > 0$

$$\int \pi(dx)\ \exp(c|x|^{1-p}) < \infty.$$

Subgeometric ergodicity for all $c > 0$ s.t. $r - 0.5c\lambda_+(1-p) > 0$

$r_*(t) \sim t^{-2p/(1+p)} \exp(\{\tilde{c}t\}^{(1-p)/(1+p)})$, $f_*(x) \sim |x|^{-2p} \exp(c|x|^{1-p})$,

and $\tilde{c} = c^{(1+p)/(1-p)}(1+p)\{r - 0.5c\lambda_+(1-p)\}$.

Example: Langevin Tempered diffusions on $\mathbb{R}^n$ (a)

Let π be a probability measure. Consider

$$\Phi_t = \Phi_0 + \int_0^t b(\Phi_s)ds + \int_0^t \sigma(\Phi_s)dB_s,$$

where

$$b(x) = \frac{1}{2}\sum_{j=1}^n a_{i,j}(x)\ \partial_{x_j} \log \pi(x) + \frac{1}{2}\sum_{j=1}^n \partial_{x_j} a_{i,j}(x) \qquad a(x) = \sigma(x)\sigma'(x).$$

Problem : π is heavy tailed

- if $\sigma(x) = ctt$, can not be geometrically ergodic Roberts Tweedie
- What happens if $\sigma(x) = \pi^{-d}(x)$, $d > 0$?

Example: Langevin Tempered diffusions on $\mathbb{R}^n$ (b)

Conditions

On π

- π is positive and C^2 on $\mathbb{R}^n$.
- π is polynomially decreasing in the tails : $\exists 0 < \beta < 1/n$

$$0 < \liminf_{|x|\to\infty} \frac{|\nabla \log \pi(x)|}{\pi^\beta(x)} \leq \limsup_{|x|\to\infty} \frac{|\nabla \log \pi(x)|}{\pi^\beta(x)} < \infty,$$

$$2\beta - 1 < \gamma := \liminf_{|x|\to\infty} \frac{\mathrm{Tr}(\nabla^2 \log \pi(x))}{|\nabla \log \pi(x)|^2} \leq \limsup_{|x|\to\infty} \frac{\mathrm{Tr}(\nabla^2 \log \pi(x))}{|\nabla \log \pi(x)|^2} < \infty,$$

On d

- $d \in \mathcal{D}_n$ s.t. the process is non-explosive.

Example: $\pi(x) \sim |x|^{-1/\beta}$

$\mathcal{D}_1 = [0; (1+\beta)/2]$ $\quad$ $\mathcal{D}_n = [0; (1+\beta(2-n))/2];$ $\quad$ $\gamma = \beta(2-n) > 2\beta - 1.$

Example: Langevin Tempered diffusions on $\mathbb{R}^n$ (c)

Itô's formula yields

$$\mathcal{A}V(x) = \langle b(x), \nabla V(x) \rangle + \frac{1}{2}\mathrm{Tr}\left(\nabla^2 V(x)\ a(x)\right).$$

Theorem (Ergodicity of the Langevin tempered diffusion) If $d \in \mathcal{D}_n$ and

$0 \le d < \beta$: Fails to be geometrically ergodic.
Is polynomially ergodic, for all $0 \le \kappa < 1 + \gamma - 2\beta$

$$\lim_t (1+t)^\tau \sup_{|g| \le 1+\pi^{-\kappa}} |P^t g(x) - \pi(g)| = 0, \qquad 0 \le \tau < \frac{1+\gamma-2\beta-\kappa}{2(\beta-d)}.$$

$0 < \beta \le d < (1+\gamma)/2$: Is geometrically ergodic, for all $0 < \kappa < 1 + \gamma - 2d$,

$$\exists \tau > 1 \qquad \lim_t \tau^t \sup_{|g| \le 1+\pi^{-\kappa}} |P^t g(x) - \pi(g)| = 0.$$

$0 < \beta < d$: Is uniformly ergodic,

$$\exists \tau > 1 \text{ and } c \qquad \forall x \in \mathsf{X}, \qquad \lim_t \tau^t \sup_{|g| \le 1} |P^t g(x) - \pi(g)| \le c.$$

Conclusion

Other results

When

$$s \mapsto V(\Phi_s) - V(\Phi_0) - \int_0^s \{-\phi \circ V(\Phi_u) + b\mathbb{I}_C(u)\} du \quad \mathbb{P}_x\text{-supermartingale},$$

$$\sup_C V < \infty$$

where C closed petite set, $V \geq 1$ cadlag, $\phi \uparrow$ differentiable concave.

Petite set

- the level sets $\{V \leq n\}$ are petite.

(f, r)-regularity

- the level sets $\{V \leq n\}$ are (f, r)-regular.
- there exists a full set, union of (f, r)-regular sets.

Skeleton

- Subgeometric moment of the return-time to a small set, for any skeleton.

Resolvent

- Subgeometric condition for any resolvent kernel.

Moderate deviation principle $\cdots$

Conclusion

Details can be found in:

- G. Fort & G.O. Roberts. Subgeometric ergodicity of strong Markov processes. *Ann. Appl. Probab. 15(2):1565-1589, 2005.*
- R. Douc, G. Fort & A. Guillin. Subgeometric rates of convergence of f-ergodic strong Markov processes. *ArXiv math.ST/0605791, 2006.*