Perfect simulation:

A (short) survey.

Wilfrid Kendall

w.s.kendall@warwick.ac.uk

Department of Statistics, University of Warwick

CRiSM workshop 22-24 August 2006:
“New Developments in MCMC”

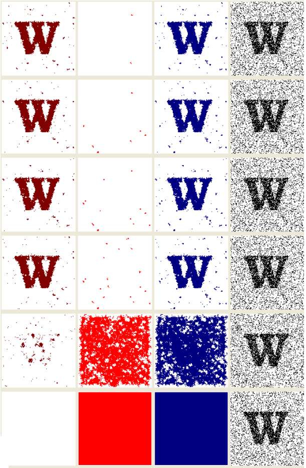
Plan of talk

Setting the scene (I)

- Propp and Wilson (1996) (and precursors);
- modify favourable *MCMC* algorithms to be exact;
- use **coupling**;
- resulting algorithms have random run-times;

Dyer and Greenhill (1999)'s Disconcerting Observation,

Huber's Rejoinder.



Setting the scene (II)

Successful *CFTP* needs "secret plans and clever tricks":

- search for monotonicity;
- crossover (WSK 1998);
- small sets (Murdoch and Green 1998);
- bounding chains (Häggström and Nelander 1999; Huber 2003);
- finitary *CFTP* (space as well as time) (WSK 1997, Häggström and Steif 2000);
- multi-shift (Wilson 2000b);
- read-once (Wilson 2000a);
- *FMMR* (Fill, Machida, Murdoch, and Rosenthal 2000).

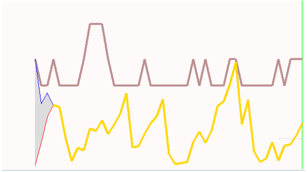
Impractical *CFTP* (I)

When is *CFTP* possible *in principle*?

- Foss and Tweedie (1998): classic *CFTP* is equivalent to uniform ergodicity

  (small-set *CFTP*, sub-sampling).
- Nevertheless a notion of "dominated *CFTP*" can be made to work in some cases of geometric ergodicity (WSK 1998, WSK and Møller 1999).

Application to *perpetuities* ($X_{n+1} = U_{n+1}^{\alpha}(1 + X_n)$): exp($M/D/1$ workload) dominates.



Algorithm performance is even better for 64-bit computing . . .

Impractical *CFTP* (II)

WSK (2004): any geometrically ergodic Markov chain X can (in principle) be adapted to *domCFTP*.

Sketch:

- Geometric ergodicity yields Foster-Lyapunov condition $\mathbb{E}\left[V(X_{n+1})|X_n\right] \leq \gamma V(X_n) + b\,\mathbb{I}\left[C\right](X_n)$.
- Markov's inequality: domination by *exp*$(D/M/1$ workload).
- Need $\gamma < e^{-1}$ to make workload positive-recurrent!
- Sub-sample to improve Foster-Lyapunov γ.
- Domination maintained even under regeneration / non-regeneration at small set C: (argument of transportation type, eg Roberts and Rosenthal 2001).

Impractical, but similar to perpetuity example.

Beyond Geometric Ergodicity (I)

(Joint work with Stephen Connor)

Consider polynomial ergodicity, eg $\mathbb{E}\left[V(X_{n+1})|X_n\right] \leq V(X_n) - V(X_n)^\alpha + b\,\mathbb{I}\left[C\right](X_n)$.

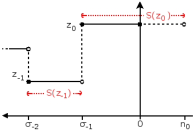
- Previous approach cannot work *in general*.
- Many Markov chains are slow because of "slow-down" in extremities of state-space;
- so use adaptive sub-sampling: $\sigma_{n+1} = \sigma_n + \lceil \lambda V(X_{\sigma_n})^\delta \rceil$;
- X is tame if (suitably) geometrically ergodic under σ-sub-sampling.

Beyond Geometric Ergodicity (II)

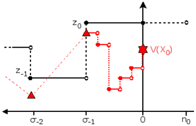
Idea: σ-time-changed X has *domCFTP* so "undo" time-change.

Construct stationary dominating process D which delivers $\exp(D/M/1 \text{ workload})$ dominator under σ-time-change.

Build X dominated by D by coupling only at times when D changes;



Careful conditional probability to show how to draw from X_0;



Proceed as with geometric ergodicity.

Beyond Geometric Ergodicity (III)

Examples of non-geometrically-ergodic tame chains:

- "Epoch chain": spend random length of time at random level before jumping to 0 and regenerating;
- Delayed death processes;
- Delayed simple random walks;
- Random walks on half-lines (Tuominen and Tweedie 1994, Jarner and Roberts 2002).

Actual examples often permit direct domination (hence *domCFTP*) *without* adaptive σ-sub-sampling.

To polynomial ergodicity and beyond!

Are all polynomially ergodic chains tame?

Sub-geometric ergodicity and *CFTP*?

Positive recurrence and *CFTP*?

Sub-sampling *versus* maximal coupling?

Questions?

Bibliography

This is a rich hypertext bibliography. Journals are linked to their homepages, and stable URL links (as provided for example by JSTOR or Project Euclid) have been added where known. Access to such URLs is not universal: in case of difficulty you should check whether you are registered (directly or indirectly) with the relevant provider. In the case of preprints, icons , , , linking to homepage locations are inserted where available: note that these are probably less stable than journal links!.

Dyer, M. and C. Greenhill (1999).
Random walks on combinatorial objects.
In *Surveys in combinatorics, 1999 (Canterbury)*, Volume 267 of *London Math. Soc. Lecture Note Ser.*, pp. 101–136. Cambridge: Cambridge University Press.

Fill, J. A., M. Machida, D. J. Murdoch, and J. S. Rosenthal (2000).
Extension of Fill's perfect rejection sampling algorithm to general chains.
Random Structures and Algorithms *17*(3-4), 290–316, doi>.

Foss, S. G. and R. L. Tweedie (1998).
Perfect simulation and backward coupling.
Stochastic Models *14*, 187–203.

Häggström, O. and K. Nelander (1999).
On exact simulation of Markov random fields using coupling from the past.
Scandinavian Journal of Statistics Theory and Applications *26*(3), 395–411.

Häggström, O. and J. E. Steif (2000).
Propp-Wilson algorithms and finitary codings for high noise Markov random fields.
Combin. Probab. Computing 9, 425–439, .

Huber, M. (2003).
A bounding chain for Swendsen-Wang.
Random Structures and Algorithms 22(1), 43–59, doi>.

Jarner, S. F. and G. O. Roberts (2002).
Polynomial convergence rates of Markov chains.
The Annals of Applied Probability 12(1), 224–247.

Kendall, W. S. (1997).
Perfect simulation for spatial point processes.
In *Bulletin ISI, 51st session proceedings, Istanbul (August 1997)*, Volume 3, pp. 163–166. Also: University of Warwick Department of Statistics Research Report 308. .

Kendall, W. S. (1998).
Perfect simulation for the area-interaction point process.
In L. Accardi and C. C. Heyde (Eds.), *Probability Towards 2000*, New York, pp. 218–234. Springer-Verlag, Also: University of Warwick Department of Statistics Research Report 292. .

THE UNIVERSITY OF WARWICK

Kendall, W. S. (2004).
Geometric ergodicity and perfect simulation.
Electronic Communications in Probability 9, 140–151, Also University of Warwick Department of Statistics Research Report 427, and http://arxiv.org/abs/math/0410012 arXiv

Kendall, W. S. and J. Møller (1999).
Perfect implementation of a Metropolis-Hastings simulation of Markov point processes.
Research Report 348, University of Warwick Department of Statistics, University of Warwick, .

Murdoch, D. J. and P. J. Green (1998).
Exact sampling from a continuous state space.
Scandinavian Journal of Statistics Theory and Applications 25, 483–502.

Propp, J. G. and D. B. Wilson (1996).
Exact sampling with coupled Markov chains and applications to statistical mechanics.
Random Structures and Algorithms 9, 223–252, doi>.

Roberts, G. O. and J. S. Rosenthal (2001).
Small and pseudo-small sets for Markov chains.
Stochastic Models 17(2), 121–145.

Tuominen, P. and R. L. Tweedie (1994).
Subgeometric rates of convergence of f-ergodic Markov chains.
Advances in Applied Probability 26(3), 775–798.

Wilson, D. B. (2000a).
How to couple from the past using a read-once source of randomness.
Random Structures and Algorithms 16(1), 85–113, doi>.

Wilson, D. B. (2000b).
Layered Multishift Coupling for use in Perfect Sampling Algorithms (with a primer on CFTP).
In N. Madras (Ed.), *Monte Carlo Methods*, Volume 26 of *Fields Institute Communications*, pp. 143–179. American Mathematical Society.