CONDITIONING AN ADDITIVE FUNCTIONAL OF A MARKOV CHAIN TO STAY NON-NEGATIVE I:SURVIVAL FOR A LONG TIME [1]

Saul D. Jacka, University of Warwick
Zorana Lazic, University of Warwick
Jon Warren, University of Warwick

Abstract

Let $(X_t)_{t\geq 0}$ be a continuous-time irreducible Markov chain on a finite statespace E, let v be a map $v : E \to \mathbb{R}\backslash\{0\}$ and let $(\varphi_t)_{t\geq 0}$ be an additive functional defined by $\varphi_t = \int_0^t v(X_s)ds$. We consider the cases where the process $(\varphi_t)_{t\geq 0}$ is oscillating and where $(\varphi_t)_{t\geq 0}$ has a negative drift. In each of the cases we condition the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that $(\varphi_t)_{t\geq 0}$ stays non-negative until time T and prove weak convergence of the conditioned process as $T \to \infty$.

1 Introduction

The problem of conditioning a stochastic process to stay forever in a certain region has been extensively studied in the literature. Many authors have addressed essentially the same problem by conditioning a process with a possibly finite lifetime to live forever. An interesting case is when the event that the process remains in some region is of zero probability, or in terms of the lifetime of the process restricted to the region, when the process has a finite lifetime with probability one. In that case the process cannot be conditioned to stay in the region forever in the standard way. Instead, this condtioning can be approximated by conditioning the process to stay in the region for a large time.

There are many well-known examples of such conditionings in which weak convergence of the approximating process occurs. For instance, Knight (1969) showed that the standard Brownian motion conditioned not to cross zero for a large time converges weakly to a three-dimensional Bessel process; Iglehart (1974) considered a general random walk conditioned to stay non-negative for a large time and showed that it converges weakly; Pinsky (1985) showed that under certain conditions, a homogeneous diffusion on $\mathbb{R}^d$ conditioned to remain in an open connected bounded region for a large time

[1]MSC Classification: Primary 60J27, Secondary 60B10
Keywords: Markov chain, conditional law, weak convergence

converges weakly to a homogeneous diffusion; Jacka and Roberts (1988) proved weak convergence of an Ito diffusion conditioned to remain in an interval (a, b) until a large time.

However, weak convergence of the approximations does not always occur. There are counterexamples in which a process conditioned to stay in a region for a large time does not converge at all or it does converge but to a dishonest limit. Bertoin and Doney (1994) and Jacka and Warren (2002) gave examples of such processes.

This paper is concerned with another example of conditioning a process to stay in a region. We consider a finite statespace continuous time Markov chain $(X_t)_{t\geq 0}$ and an associated fluctuating additive functional $(\varphi_t)_{t\geq 0}$. The aim is to condition the Markov process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the fluctuating functional stays non-negative.

There are three possible cases of the behaviour of the process $(\varphi_t)_{t\geq 0}$, in two of which, when it oscillates and when it drifts to $-\infty$, the event that it stays non-negative is of zero probability. We are interested in performing conditioning in these two cases.

A similar question has been discussed in Bertoin and Doney (1994) for a real-valued random walk. It has been shown there that, under certain conditions, an oscillating random walk or a random walk with a negative drift, conditioned to stay non-negative for large time converges weakly to an honest limit which is an h-transform of the original random walk killed when it hits zero. This work presents the analogous result for the process $(X_t, \varphi_t)_{t\geq 0}$.

The organisation of the paper is as follows: the exact formulation of the problem and the main results are given in Section 2, the notation and preliminary results used in the proofs of the main theorems are given in Section 3, the proof of the result in the oscillating case is given is Section 4 and the proof of the result in the negative drift case is given in Section 5.

2 The problem and main results

Let $(X_t)_{t\geq 0}$ be an irreducible honest Markov chain on a finite statespace E. Let v be a map $v : E \to \mathbb{R}\backslash\{0\}$ and suppose that both $E^+ = v^{-1}(0,\infty)$ and $E^- = v^{-1}(-\infty, 0)$ are non-empty.

Define the process $(\varphi_t)_{t\geq 0}$ by

$$\varphi_t = \varphi + \int_0^t v(X_s)ds, \quad \varphi \in \mathbb{R}.$$

Let, for any $y \in \mathbb{R}$, $E_y^+ = (E \times (y, +\infty)) \bigcup (E^+ \times \{y\})$ and let $H_0 = \inf\{t > 0 : \varphi_t < 0\}$. The aim is to condition the process $(X_t, \varphi_t)_{t\geq 0}$ starting in E_0^+ on the event $\{H_0 = +\infty\}$.

There are three possible cases depending on the behaviour of the process $(\varphi_t)_{t\geq 0}$. When the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$, the event $\{H_0 = +\infty\}$ is of positive probablity which implies that conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on it can be performed in the standard way. However, when the process $(\varphi_t)_{t\geq 0}$ oscillates or drifts to $-\infty$, the event $\{H_0 = +\infty\}$ is of zero probability and conditioning $(X_t, \varphi_t)_{t\geq 0}$ on it cannot be performed in the standard way. We concentrate on these two latter cases and define conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_0 = +\infty\}$ as the limit as $T \to \infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_0 > t\}$.

Let $P_{(e,\varphi)}$ denote the law of the process $(X_t, \varphi_t)_{t\geq 0}$ starting at (e, φ) and let $E_{(e,\varphi)}$ denote the expectation operator associated with $P_{(e,\varphi)}$. Let $P^{(T)}_{(e,\varphi)}$, $T > 0$, denote the law of the process $(X_t, \varphi_t)_{t\geq 0}$, starting at $(e, \varphi) \in E_0^+$, conditioned on $\{H_0 > T\}$, and let $P^{(T)}_{(e,\varphi)}|_{\mathcal{F}_t}$, $t \geq 0$, be the restriction of $P^{(T)}_{(e,\varphi)}$ to $\mathcal{F}_t$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(X_t)_{t\geq 0}$. We are interested in weak convergence of $(P^{(T)}_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $T \to +\infty$.

Let Q denote the conservative irreducible Q-matrix of the process $(X_t)_{t\geq 0}$ and let V be the diagonal matrix $diag(v(e))$. Let $V^{-1}Q\Gamma = \Gamma G$ be the unique Wiener-Hopf factorisation of the matrix $V^{-1}Q$ (see Barlow et al. (1980) or refer to Lemma 3.4 below). Let J, J_1 and J_2 be the matrices

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad J_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and let a matrix Γ_2 be given by $\Gamma_2 = J\Gamma J$.

Now we state our main result in the oscillating case.

Theorem 2.1 *Suppose that the process $(\varphi_t)_{t\geq 0}$ oscillates. Then, for fixed $(e, \varphi) \in E_0^+$ and $t \geq 0$, the measures $(P^{(T)}_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ converge weakly as $T \to \infty$ to a probability measure $P^{h_r}_{(e,\varphi)}|_{\mathcal{F}_t}$ defined by*

$$P^{h_r}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)h_r(X_t, \varphi_t)I\{t < H_0\}\Big)}{h_r(e, \varphi)}, \qquad t \geq 0,\ A \in \mathcal{F}_t,$$

where $h_r(e, y)$ is a positive harmonic function for the process $(X_t, \varphi_t)_{t\geq 0}$ given by $h_r(e, y) = e^{-yV^{-1}Q}J_1\Gamma_2 r(e)$, $(e, y) \in E \times \mathbb{R}$, and $V^{-1}Qr = 1$.

Let β_0 be the point at which the Perron-Frobenius eigenvalue $\alpha(\beta)$ of the matrix $(Q - \beta V)$ attains its global minimum (see Lemma 3.9 below). Let $\alpha_0 = \alpha(\beta_0)$ and g_0 be the Perron-Frobenius eigenvalue and right eigenvector, respectively, of the matrix $(Q - \beta_0 V)$ and let G_0 be the diagonal matrix $diag(g_0(e))$. Let Q^0 be the $E \times E$ matrix with entries

$$Q^0(e, e') = G_0^{-1}(Q - \alpha_0 I - \beta_0 V)G_0(e, e'), \qquad e, e' \in E. \tag{2.1}$$

As we shall see later, the matrix Q^0 is a conservative irreducible Q-matrix (see Lemma 3.11 below). Let $(V^{-1}Q)\Gamma^0 = \Gamma^0 G^0$ be the unique Wiener-Hopf factorization of the matrix $V^{-1}Q^0$ and let $\Gamma_2^0 = J\Gamma^0 J$. Now we can state our main result in the negative drift case.

Theorem 2.2 *Suppose that the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$. For fixed $(e,\varphi),(e',\varphi') \in E_0^+$ and $t \geq 0$, if all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple and if $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$ exists, then the measures $(P_{(e,\varphi)}^{(T)}|_{\mathcal{F}_t})_{T\geq 0}$ converge weakly as $T \to \infty$ to a probability measure $P_{(e,\varphi)}^{h_{r^0}}|_{\mathcal{F}_t}$ which is defined by*

$$P_{(e,\varphi)}^{h_{r^0}}(A) = \frac{E_{(e,\varphi)}\Big(I(A)h_{r^0}(X_t,\varphi_t,t)I\{t<H_0\}\Big)}{h_{r^0}(e,\varphi,t)}, \qquad t\geq 0,\ A\in\mathcal{F}_t,$$

where the function $h_{r^0}(e,y,t)$ is positive and space-time harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and is given by $h_{r^0}(e,y,t) = e^{-\alpha_0 t}e^{-\beta_0 y}\ G_0\ e^{-yV^{-1}Q^0}J_1\Gamma_2^0 r^0(e)$, $(e,y,t) \in E\times\mathbb{R}\times[0,+\infty)$, and $V^{-1}Q^0 r^0 = 1$.

Note that $P_{(e,\varphi)}^{h_r}$ and $P_{(e,\varphi)}^{h_{r^0}}$ are h-transforms of the transition kernel for the process $(X_t,\varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero.

3 Notation and preliminary results

The purpose of this section is to introduce notation and recall some results and prove some others which are needed for the proofs in the subsequent two sections. The proofs are fairly straightforward and are included for the sake of completeness.

Lemma 3.1 *Let Q be an irreducible essentially non-negative matrix, V a diagonal matrix and $\beta\in\mathbb{R}$. Then the matrix $(Q-\beta V)$ is also an irreducible essentially non-negative matrix.*

Proof: The proof follows directly from the definition of an irreducible essentially non-negative matrix (see Seneta (1981)). □

The following three lemmas were proved in Barlow *et al.* (1980). We state them here in the notation we are going to use.

Lemma 3.2 *For fixed $\alpha > 0$, there exists a unique pair $(\Pi_\alpha^+,\Pi_\alpha^-)$, where Π_α^+ is an $E^-\times E^+$ matrix and Π_α^- is an $E^+\times E^-$ matrix, and there exist Q-matrices G_α^+ and G_α^- on $E^+\times E^+$ and $E^-\times E^-$, respectively, such that, if*

$$\Gamma_\alpha = \begin{pmatrix} I & \Pi_\alpha^- \\ \Pi_\alpha^+ & I\end{pmatrix} \quad \text{and} \quad G_\alpha = \begin{pmatrix} G_\alpha^+ & 0 \\ 0 & -G_\alpha^-\end{pmatrix},$$

then Γ_α *is invertible and* $\Gamma_\alpha^{-1}\, V^{-1}(Q-\alpha I)\, \Gamma_\alpha = G_\alpha$. *Moreover,* Π_α^+ *and* Π_α^- *are strictly substochastic.*

We recall that the subspace E_y^+, $y \in \mathbb{R}$, is given by $E_y^+ = (E\times(y,+\infty)) \bigcup (E^+\times\{y\})$. Let E_y^-, $y \in \mathbb{R}$, be the subspace $E_y^- = (E\times(-\infty,y)) \bigcup (E^-\times\{y\})$. Let H_y, $y \in \mathbb{R}$, be the first crossing time of the level y by the process $(\varphi_t)_{t\geq 0}$ defined by

$$H_y = \begin{cases} \inf\{t>0 : \varphi_t < y\} & \text{if } \ (X_t,\varphi_t)_{t\geq 0} \text{ starts in } E_y^+ \\ \inf\{t>0 : \varphi_t > y\} & \text{if } \ (X_t,\varphi_t)_{t\geq 0} \text{ starts in } E_y^-. \end{cases}$$

Lemma 3.3 *Let* $\alpha > 0$ *be fixed. Then*

$$\begin{array}{llll} E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0}=e'\}) & = \Pi_\alpha^+(e,e'), & (e,e')\in E^-\times E^+, & \\ E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0}=e'\}) & = \Pi_\alpha^-(e,e'), & (e,e')\in E^+\times E^-, & \\ E_{(e,0)}(e^{-\alpha H_y} I\{X_{H_y}=e'\}) & = e^{yG_\alpha^+}(e,e'), & (e,e')\in E^+\times E^+, & y>0, \\ E_{(e,0)}(e^{-\alpha H_{-y}} I\{X_{H_{-y}}=e'\}) & = e^{yG_\alpha^-}(e,e'), & (e,e')\in E^-\times E^-, & y>0. \end{array}$$

Lemma 3.4 *There exists a unique pair* (Π^+,Π^-), *where* Π^+ *is an* $E^-\times E^+$ *matrix and* Π^- *is an* $E^+\times E^-$ *matrix, and there exist Q-matrices* G^+ *on* $E^+\times E^+$ *and* G^- *on* $E^-\times E^-$ *such that*

$$(V^{-1}Q)\,\Gamma = \Gamma\, G, \tag{3.2}$$

where

$$\Gamma = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \quad \textit{and} \quad G = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}.$$

Moreover, Π^+ *and* Π^- *are substochastic and*

$$\begin{array}{llll} P_{(e,0)}(X_{H_0}=e') & = \Pi^+(e,e'), & (e,e')\in E^-\times E^+, & \\ P_{(e,0)}(X_{H_0}=e') & = \Pi^-(e,e'), & (e,e')\in E^+\times E^-, & \\ P_{(e,0)}(X_{H_y}=e') & = e^{yG^+}(e,e'), & (e,e')\in E^+\times E^+, & y\geq 0, \\ P_{(e,0)}(X_{H_{-y}}=e') & = e^{yG^-}(e,e'), & (e,e')\in E^-\times E^-, & y\geq 0. \end{array}$$

Lemmas 3.2 and 3.4 are said to yield the Wiener-Hopf factorizations of the matrices $V^{-1}(Q-\alpha I)$, $\alpha>0$, and $V^{-1}Q$, respectively.

The statements in the following lemma can easily be deduced from Lemmas 3.2 - 3.4, thus we omit their proofs.

Lemma 3.5 *(i) The matrices* Π^+ *and* Π^- *are positive.*

(ii) If at least one of the matrices Π^+ *and* Π^- *is strictly substochastic then the matrices* $(I-\Pi^-\Pi^+)$, $(I-\Pi^+\Pi^-)$ *and* Γ *are invertible and*

$$\Gamma^{-1} = \begin{pmatrix} (I-\Pi^-\Pi^+)^{-1} & -\Pi^-(I-\Pi^+\Pi^-)^{-1} \\ -\Pi^+(I-\Pi^-\Pi^+)^{-1} & (I-\Pi^+\Pi^-)^{-1} \end{pmatrix}.$$

(iii) *The matrices* G^+ *and* G^- *are irreducible* Q*-matrices.*

(iv) G^+ *(*G^-*) is conservative iff* Π^+ *(*Π^-*) is stochastic.*

(v) $\lim_{\alpha\to 0}\Gamma_\alpha = \Gamma$.

(vi) *For any* $y > 0$ *and* $(e,\varphi) \in E_0^+ \cap E_y^-$*, or any* $y < 0$ *and* $(e,\varphi) \in E_0^- \cap E_y^+$*,* $P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0) > 0$ *and* $0 < P_{(e,\varphi)}(H_y < H_0) < 1$.

(vii) *For any* $(e,\varphi) \in E_0^+$ *and* $e' \in E^-$*, or any* $(e,\varphi) \in E_0^-$ *and* $e' \in E^+$*,* $P_{(e,\varphi)}(X_{H_0} = e', H_0 < +\infty) > 0$.

(viii) *For any* $(e,\varphi) \in E \times \mathbb{R}$ *and* $T > 0$*,* $P_{(e,\varphi)}(H_0 > T) > 0$.

We introduce vector notation that will be in use in the sequel. For any vector g on E, let g^+ and g^- denote its restrictions to E^+ and E^- respectively. We write the column vector g as $g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix}$ and the row vector μ as $\mu = (\,\mu^+ \quad \mu^-\,)$.

It follows from Lemmas 3.2 - 3.4 (see Barlow *et al.* (1980)) that the matrix $V^{-1}(Q-\alpha I)$ cannot have strictly imaginary eigenvalues and there exists a basis $\mathcal{B}(\alpha)$ in the space of all vectors on E such that if $g(\alpha)$ is in $\mathcal{B}(\alpha)$, then

$$(V^{-1}(Q-\alpha I) - \lambda(\alpha)\; I)^k g(\alpha) = 0, \tag{3.3}$$

for some eigenvalue $\lambda(\alpha)$ of $V^{-1}(Q-\alpha I)$ and some $k \in \mathbb{N}$. The number of vectors in the basis $\mathcal{B}(\alpha)$ associated with the same eigenvalue is equal to the algebraic multiplicity of that eigenvalue. Let $\mathcal{N}(\alpha)$ and $\mathcal{P}(\alpha)$ be the sets of vectors $g(\alpha) \in \mathcal{B}(\alpha)$ associated with eigenvalues with positive and with negative real parts, respectively.

Then,

$$g(\alpha) \in \mathcal{N}(\alpha) \Rightarrow g(\alpha) = \begin{pmatrix} g^+(\alpha) \\ \Pi_\alpha^+ g^+(\alpha) \end{pmatrix}, \quad g(\alpha) \in \mathcal{P}(\alpha) \Rightarrow g(\alpha) = \begin{pmatrix} \Pi_\alpha^- g^-(\alpha) \\ g^-(\alpha) \end{pmatrix}. \tag{3.4}$$

The set $\mathcal{N}(\alpha)$ (respectively $\mathcal{P}(\alpha)$) contains exactly $|E^+|$ (respectively E^-) vectors and the vectors $g^+(\alpha)$ (respectively $g^-(\alpha)$) for all $g(\alpha) \in \mathcal{N}(\alpha)$ (respectively $\mathcal{P}(\alpha)$) form a basis in the space of all vectors on E^+ (respectively E^-). The eigenvalues of $V^{-1}(Q-\alpha I)$ with strictly negative (respectively positive) real part coincide with the eigenvalues of G_α^+ (respectively $-G_\alpha^-$).

The Wiener-Hopf factorization (3.2) of the matrix $V^{-1}Q$ implies that

$$\begin{aligned} G^+ f^+ = \alpha f^+ \quad &\text{iff} \quad V^{-1}Q \begin{pmatrix} f^+ \\ \Pi^+ f^+ \end{pmatrix} = \alpha \begin{pmatrix} f^+ \\ \Pi^+ f^+ \end{pmatrix} \\ G^- g^- = -\beta g^- \quad &\text{iff} \quad V^{-1}Q \begin{pmatrix} \Pi^- g^- \\ g^- \end{pmatrix} = \beta \begin{pmatrix} \Pi^- g^- \\ g^- \end{pmatrix}. \end{aligned} \tag{3.5}$$

Let α_j, $j = 1, \ldots, n$, be the eigenvalues (not necessarily distinct) of the matrix G^+, and $-\beta_k$, $k = 1, \ldots, m$, be the eigenvalues (not necessarily distinct) of the matrix G^-. By Lemma 3.5 (iii) G^+ and G^- are irreducible Q-matrices, which implies that

$$\alpha_{max} \equiv \max_{1 \leq j \leq n} Re(\alpha_j) \leq 0 \quad \text{and} \quad -\beta_{min} \equiv \max_{1 \leq k \leq m} Re(-\beta_k) = -\min_{1 \leq k \leq m} Re(\beta_k) \leq 0$$

are simple eigenvalues of G^+ and G^-, respectively. Hence, it follows from (3.5) that all eigenvalues of $V^{-1}Q$ with negative (respectively positive) real part coincide with the eigenvalues of G^+ (respectively $-G^-$).

By Jordan normal form theory there exists a basis $\mathcal{B}$ in the space of all vectors on E such that there exist exactly $n = |E^+|$ vectors $\{f_1, f_2, \ldots, f_n\}$ in $\mathcal{B}$ such that each vector f_j, $j = 1, \ldots, n$ is associated with an eigenvalue α_j of $V^{-1}Q$ for which $Re(\alpha_j) \leq 0$, and that there exist exactly $m = |E^-|$ vectors $\{g_1, g_2, \ldots, g_m\}$ in $\mathcal{B}$ such that each vector g_k, $k = 1, \ldots, m$, is associated with an eigenvalue β_k of $V^{-1}Q$ with $Re(\beta_k) \geq 0$. The vectors $\{f_1^+, f_2^+, \ldots, f_n^+\}$ form a basis $\mathcal{N}^+$ in the space of all vectors on E^+. and the vectors $\{g_1^-, g_2^-, \ldots, g_m^-\}$ form a basis $\mathcal{P}^-$ in the space of all vectors on E^-.

Let f_{max} and g_{min} be the eigenvectors of the matrix $V^{-1}Q$ associated with its eigenvalues α_{max} and β_{min}, respectively. Then, f_{max}^+ and g_{min}^- are the Perron-Frobenius eigenvectors of the matrices G^+ and G^-, respectively.

Lemma 3.6 *(i) The vectors f_{max} and g_{min} are the only positive eigenvectors of the matrix $V^{-1}Q$.*

(ii) There are no non-negative vectors on E^+ (E^-) which are linearly independent of the vector f_{max}^+ (g_{min}^-).

Proof: (i) Let f be a positive eigenvector of the matrix $V^{-1}Q$. Then, by (3.5), either f^+ is an eigenvector of G^+ or f^- is an eigenvector of G^-. The only positive eigenvectors of G^+ and G^- are f_{max}^+ and g_{min}^-, respectively. Hence,

$$f = \begin{pmatrix} f_{max}^+ \\ \Pi^+ f_{max}^+ \end{pmatrix} = f_{max} \quad \text{or} \quad f = \begin{pmatrix} \Pi^- g_{min}^- \\ g_{min}^- \end{pmatrix} = g_{min}.$$

Since, by Lemma 3.5 (i), the matrices Π^+ and Π^- are positive, we have that f_{max} and g_{min} are positive which completes the proof.

(ii) Let f^+ be a non-negative vector on E^+ independent of f_{max}^+. Since $\mathcal{N}^+ = \{f_1^+, f_2^+, \ldots, f_n^+\}$ is a basis in the space of all vectors on E^+, the vector f^+ has a decomposition

$$f^+ = \sum_{f_j^+ \neq f_{max}^+} a_j \, f_j^+,$$

for some coefficients a_j, $j = 1, \ldots, n$.

Let $f^{left,+}_{max}$ be the left Perron-Frobenius eigenvector of G^+. Then $f^{left,+}_{max} e^{tG^+} = e^{\alpha_{max} t} f^{left,+}_{max}$ and $f^{left,+}_{max} f^+_j = 0$ for all $f^+_j \neq f^+_{max}$, $j = 1, \ldots, n$. Thus, for any $t \geq 0$,

$$f^{left,+}_{max} e^{tG^+} f^+ = \sum_{f^+_j \neq f^+_{max}} a_j \, f^{left,+}_{max} e^{tG^+} f^+_j = \sum_{f^+_j \neq f^+_{max}} a_j \, e^{\alpha_j t} f^{left,+}_{max} f^+_j = 0,$$

but that is a contradiction because f^+ and $f^{left,+}_{max}$ are non-negative and $f^{left,+}_{max} e^{tG^+} f^+ = e^{\alpha_{max} t} f^{left,+}_{max} f^+ > 0$. Therefore, the vectors f^+ and f^+_{max} are not linearly independent.

□

Let a matrix $F(y)$, $y \in \mathbb{R}$, be defined by

$$F(y) \quad = \quad \begin{cases} J_1 \, e^{yG} = e^{yG} \, J_1, & y > 0 \\ J_2 \, e^{yG} = e^{yG} \, J_2, & y < 0. \end{cases}$$

Then

Lemma 3.7 *For any $e, e' \in E$,*

$$P_{(e,\varphi)}(X_{H_0} = e', H_0 < +\infty) = \Gamma \; F(-\varphi)(e,e'), \quad \varphi \neq 0,$$
$$P_{(e,0)}(X_{H_0} = e', H_0 < +\infty) = (I - \Gamma_2)(e,e') = \begin{pmatrix} 0 & \Pi^- \\ \Pi^+ & 0 \end{pmatrix} (e,e').$$

Proof: The lemma follows directly from the definition of the matrices Γ, Γ_2 and $F(\varphi)$.

□

Let $\mathcal{G}$ be the infinitesimal generator of the process $(X_t, \varphi_t)_{t \geq 0}$ and let $\mathcal{D}_{\mathcal{G}}$ denote its domain. Let a function $f(e,\varphi)$ on $E \times \mathbb{R}$ be continuously differentiable in φ. Then $f \in \mathcal{D}_{\mathcal{G}}$ and

$$\mathcal{G} f = \Big(Q + V \frac{\partial}{\partial \varphi} \Big) f, \tag{3.6}$$

where

$$\begin{array}{rcl} Qf(e,\varphi,t) & = & \sum_{e' \in E} Q(e,e') f(e',\varphi) \\ V \frac{\partial f}{\partial \varphi}(e,\varphi,t) & = & V(e,e) \frac{\partial f}{\partial \varphi}(e,\varphi). \end{array}$$

Similarly, let $\mathcal{A}$ be the infinitesimal generator of the process $(X_t, \varphi_t, t)_{t \geq 0}$ and let $\mathcal{D}_{\mathcal{A}}$ denote its domain. Let a function $f(e,\varphi,t)$ on $E \times \mathbb{R} \times [0,+\infty)$ be continuously differentiable in φ and t. Then $f \in \mathcal{D}_{\mathcal{A}}$ and

$$\mathcal{A} f = \Big(Q + V \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial t} \Big) f. \tag{3.7}$$

The behaviour of the process $(\varphi_t)_{t\geq 0}$ is determined by the matrices Q and V. More precisely,

$$\begin{array}{rlll} (\varphi_t)_{t\geq 0} \text{ drifts to } +\infty & \text{iff} \;\; \mu V 1 > 0 & \text{iff} & G^+ \textit{ is conservative}, G^- \textit{ is not conservative}, \\ (\varphi_t)_{t\geq 0} \text{ oscillates} & \text{iff} \;\; \mu V 1 = 0 & \text{iff} & G^+ \textit{ and } G^- \textit{ are conservative}, \\ (\varphi_t)_{t\geq 0} \text{ drifts to } -\infty & \text{iff} \;\; \mu V 1 < 0 & \text{iff} & G^- \textit{ is conservative}, G^+ \textit{ is not conservative}. \end{array} \tag{3.8}$$

Let $f_1 = f_{max}$ and $g_1 = g_{min}$ be the eigenvectors of $V^{-1}Q$ associated with the eigenvalues α_{max} and β_{min}, respectively. Then, in the positive drift case, $f_{max} = 1 \neq g_{min}$, and in the negative drift case, $g_{min} = 1 \neq f_{max}$, and in both cases the basis $\mathcal{B}$ in the space of all vectors on E is equal to $\{f_j, j = 1, \ldots, n,\; g_k, k = 1, \ldots, m\}$. In the oscillating case, $f_{max} = g_{min} = 1$ and the equation $V^{-1}Qx = 1$ has a solution. If r is a solution, then, by Jordan normal form theory, r is linearly independent from the vectors $\{f_j, j = 1, \ldots, n,\; g_k, k = 1, \ldots, m\}$ and $\mathcal{B} = \{1, r, f_j, j = 2, \ldots, n,\; g_k, k = 2, \ldots, m\}$ is a basis in the space of all vectors on E.

The following lemmas are concerned with the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$. For any $\beta \in \mathbb{R}$, let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$ and let $u^{left}(\beta)$ and $u^{right}(\beta)$ be the associated left and right eigenvectors such that $\|u^{left}(\beta)\| = \|u^{right}(\beta)\| = 1$ in some norm in the space $\mathbb{R}^E$. A striking property of the eigenvalue $\alpha(\beta)$ is that it is a convex function of β.

Lemma 3.8 *Let $\beta \in \mathbb{R}$ and let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$. Then, $\alpha(\beta)$ is a convex function of β and therefore continuous. It attains its global minimum and has two zeros, $\alpha_{max} \leq 0$ and $\beta_{min} \geq 0$, not necessarily distinct.*

Proof: Let $r(A)$ denote the Perron-Frobenius eigenvalue of an essentially non-negative matrix A. Since Q is essentially non-negative it follows from Cohen (1981) that for any $x, y \in \mathbb{R}$ and any t, $0 < t < 1$,

$$r((1-t)(Q - xV) + t(Q - yV)) \leq (1-t)\; r(Q - xV) + t\; r(Q - yV). \tag{3.9}$$

Hence, $\alpha(\beta)$ is a convex function and therefore continuous.

Let $|\beta|$ be sufficiently large. Then some rows of $(Q - \beta V)$ are non-negative which implies that there does not exists a positive vector f such that $(Q - \beta V)f \leq 0$. Hence, by the Perron-Frobenius theorem, $\alpha(\beta) > 0$ for sufficiently large $|\beta|$.

Suppose that $\alpha(\beta) = 0$. Then there exists a positive vector f such that $(Q - \beta V)f = 0$. Since, by Lemma 3.6 (i), there exist exactly two eigenvalues of $V^{-1}Q$, α_{max} and β_{min} (not necessarily distinct), whose associated eigenvectors are positive, it follows that α_{max} and β_{min} are the only zeros of $\alpha(\beta)$.

Therefore, the function $\alpha(\beta)$ is continuous, for $|\beta|$ sufficiently large it is positive and it has either one or two zeros. All of these together imply that $\alpha(\beta)$ attains its minimum. □

Lemma 3.9 *Let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue and let $u^{left}(\beta)$ and $u^{right}(\beta)$ be the unit Perron-Frobenius left and right eigenvectors of the matrix $(Q - \beta V)$. Then $\alpha(\beta)$ is a differentiable function of β and*

$$\frac{d\alpha}{d\beta}(\beta) = -\frac{u^{left}(\beta)\ V\ u^{right}(\beta)}{u^{left}(\beta)\ u^{right}(\beta)}.$$

In addition, there is a unique $\beta_0 \in (\alpha_{max}, \beta_{min})$ such that $\frac{d\alpha}{d\beta}(\beta_0) = 0$ and $\alpha_0 \equiv \alpha(\beta_0)$ is the global minimum of the function $\alpha(\beta)$ and that

$$\frac{d\alpha}{d\beta}(\beta) \begin{cases} < 0, & \text{if } \beta < \beta_0 \\ = 0, & \text{if } \beta = \beta_0 \\ > 0, & \text{if } \beta > \beta_0, \end{cases}$$

Proof: By multiplying the equality

$$\begin{aligned}(Q - \beta V)\ u^{right}(\beta + h) - h\ V\ u^{right}(\beta + h) \\ = (\alpha(\beta + h) - \alpha(\beta))\ u^{right}(\beta + h)\ +\ \alpha(\beta)\ u^{right}(\beta + h),\end{aligned}$$

by $\frac{u^{left}(\beta)}{h}$ and by letting $h \to 0$, we obtain that $\alpha(\beta)$ is a differentiable function of β. By Lemma 3.8 it is also convex and attains its minimum. Hence, there exists unique β_0 such that $\alpha(\beta_0)$ is the global minimum of $\alpha(\beta)$ and that $\frac{d\alpha}{d\beta}(\beta_0) = 0$. By Lemma 3.8, $\alpha(\beta)$ has two zeros, $\alpha_{max} \leq 0$ and $\beta_{min} \geq 0$. Hence, $\beta_0 \in (\alpha_{max}, \beta_{min})$ when $\alpha_{max} \neq \beta_{min}$ and $\beta_0 = \alpha_{max} = \beta_{min}$ when $\alpha_{max} = \beta_{min}$.

It remains to show that $\alpha(\beta)$ is strictly monotone on $(-\infty, \beta_0]$ and $[\beta_0, +\infty)$, Let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$.

(i) Suppose that $\beta_0 = 0$. Then $\alpha(\beta_0) = 0$ and therefore $\alpha(\beta) \geq 0$. By Lemma 3.6 (i), for $\alpha > 0$, the only positive eigenvectors of $V^{-1}(Q - \alpha I)$ are $f_{max}(\alpha)$ and $g_{min}(\alpha)$ which are associated with the eigenvalues $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$, respectively. Hence, for fixed $\alpha \geq \alpha_0$, there exist only two values of β, $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$, such that α is the Perron-Frobenius eigenvalue of $(Q - \beta V)$. Since $\alpha_{max}(\alpha) \leq 0$ and $\beta_{min}(\alpha) \geq 0$, it follows that $\alpha(\beta)$ is strictly monotone on both intervals $(-\infty, 0]$ and $[0, +\infty)$.

(ii) Let now $\beta_0 \in \mathbb{R}$ and let

$$Q_0 = Q - \beta_0 V - \alpha_0 I.$$

The matrix Q_0 is essentially non-negative and, by Lemma 3.1, irreducible, and so is the matrix $(Q_0 - \beta V)$ for any $\beta \in \mathbb{R}$. Let $\alpha_0(\beta)$ be the Perron-Frobenius eigenvalue of $(Q_0 - \beta V)$. Then $\alpha_0(\beta) = \alpha(\beta + \beta_0) - \alpha_0$. Since $\alpha(\beta)$ attains its global minimum at $\beta = \beta_0$, it follows that $\alpha_0(\beta)$ attains its global minimum zero at $\beta = 0$. Therefore, by (i), $\alpha_0(\beta)$ is strictly monotone on $(-\infty, 0]$ and $[0, +\infty)$, which implies that $\alpha(\beta)$ is strictly monotone on $(-\infty, \beta_0]$ and $[\beta_0, +\infty)$. □

The sign of the unique argument β_0 of the global minimum of the function $\alpha(\beta)$, whose existence has been proved in the previous lemma, is found to depend on the behaviour of the process $(\varphi_t)_{t\geq 0}$. Namely,

Lemma 3.10

In the positive drift case	$\beta_0 > 0$ *and* $\alpha_0 < 0$.	
In the oscillating case	$\beta_0 = 0$ *and* $\alpha_0 = 0$.	
In the negative drift case	$\beta_0 < 0$ *and* $\alpha_0 < 0$.	

Proof: In the drift cases, $\alpha_{max} \neq \beta_{min}$ and therefore, by Lemma 3.9, $\beta_0 \in (\alpha_{max}, \beta_{min})$. In the positive drift case, by (3.8), $\alpha_{max} = 0$ and $\beta_{min} > 0$, and therefore $\beta_0 > 0$. In the negative drift case, by (3.8), $\beta_{min} = 0$ and $\alpha_{max} < 0$, and therefore $\beta_0 < 0$. Since in both cases the function $\alpha(\beta)$ has two distinct zeros, its global minimum α_0 is negative.

Finally, in the oscillating case, by (3.8), $\alpha_{max} = \beta_{min} = 0$ and then $\beta_0 = 0$. Thus, the function $\alpha(\beta)$ has exactly one zero at $\beta = 0$ and, since by Lemma 3.8, it attains a global minimum, it follows that $\alpha(\beta)$ attains its global minimum at $\beta_0 = 0$ and that $\alpha_0 = \alpha(\beta_0) = 0$. □

Lemma 3.11 *The matrix Q^0 given by (2.1) is a conservative irreducible Q-matrix. In addition, if μ^0 is a vector on E such that $\mu^0 Q^0 = 0$ then $\mu^0 V 1 = 0$.*

Proof: Since the matrices I and V are diagonal and the vector g_0 is positive, the matrix Q^0 is essentially non-negative. In addition, $Q^0 1 = 0$.

By Lemma 3.1, the matrix $(Q - \alpha_0 I - \beta_0 V)$ is irreducible which implies that the matrix $e^{t(Q-\alpha_0 I - \beta_0 V)}$ is positive for all $t > 0$. Since the vector g_0 is positive, it follows from the definition of Q^0 that e^{tQ^0} is positive for all $t > 0$ and that the matrix Q^0 is irreducible.

Let g_0^{left} be the left Perron-Frobenius eigenvector of the matrix $(Q - \beta_0 V)$ and let μ^0 be a vector on E with entries $\mu^0(e) = g_0^{left}(e) g_0(e)$, $e \in E$. Then $\mu^0 Q^0 = 0$ and by Lemmas 3.9 and 3.10 $\mu^0 V 1 = 0$. Since any vector v which satisfies $vQ^0 = 0$ is a constant multiple of μ^0, the proof of the lemma is complete. □

We recall the matrix $G_0 = diag(g_0)$. Since the vector g_0 is positive, the matrix G_0 is invertible.

Lemma 3.12 *For $\alpha > 0$, let*

$$V^{-1}(Q - \alpha I)\, \Gamma_\alpha = \Gamma_\alpha\, G_\alpha \quad \textit{and} \quad V^{-1}(Q^0 - \alpha I)\, \Gamma^0_\alpha = \Gamma^0_\alpha\, G^0_\alpha,$$

be the Wiener-Hopf factorisations of $V^{-1}(Q - \alpha I)$ and $V^{-1}(Q^0 - \alpha I)$, respectively. Then,

$$G^0_{\alpha-\alpha_0} = G_0^{-1}\, (G_\alpha - \beta_0 I)\, G_0, \qquad \textit{and} \quad \Gamma^0_{\alpha-\alpha_0} = G_0^{-1}\, \Gamma_\alpha\, G_0, \qquad \alpha > 0.$$

Proof: By the definition of Q^0 and by the Wiener-Hopf factorization of $V^{-1}(Q-\alpha I)$, $\alpha > 0$, given in Lemma 3.2,

$$V^{-1}(Q^0-(\alpha-\alpha_0)I) = (G_0^{-1}\Gamma_\alpha G_0)\ (G_0^{-1}(G_\alpha-\beta_0 I)G_0)\ (G_0^{-1}\Gamma_\alpha^{-1}G_0). \tag{3.10}$$

Let G_0^+ (respectively G_0^-) be the restriction of G_0 to $E^+\times E^+$ (respectively $E^-\times E^-$). Then,

$$G_0^{-1}(G_\alpha-\beta_0 I)G_0 = \begin{pmatrix} (G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+ & 0 \\ 0 & -(G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^- \end{pmatrix}.$$

Suppose that $(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+$ and $(G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^-$ are Q-matrices. Then, by Lemma 3.2, (3.10) is the the Wiener-Hopf factorization of $V^{-1}(Q^0-(\alpha-\alpha_0)I)$ for $\alpha > 0$, and by the uniqueness of the Wiener-Hopf factorization

$$G^0_{\alpha-\alpha_0} = G_0^{-1}\ (G_\alpha-\beta_0 I)\ G_0, \qquad \Gamma^0_{\alpha-\alpha_0} = G_0^{-1}\ \Gamma_\alpha\ G_0, \qquad \alpha > 0.$$

Therefore, all we have to prove is that $(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+$ and $(G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^-$ are Q-matrices.

Let the function h be defined by $h(e,\varphi,t) = e^{-\alpha_0 t}\ e^{-\beta_0\varphi}\ g_0(e)$. Then h is continuously differentiable in φ and t, and, by (3.7), it is in the domain of the infinitesimal generator $\mathcal{A}$ of the process $(X_t,\varphi_t,t)_{t\geq 0}$ and $\mathcal{A}h = 0$. It follows that the process $(h(X_{t\wedge H_y},\varphi_{t\wedge H_y},t\wedge H_y))_{t\geq 0}$ is a positive martingale. By Fatou's lemma,

$$E_{(e,\varphi)}\Big(e^{-\alpha_0 H_y}e^{-\beta_0\varphi_{H_y}}\ g_0(X_{H_y})\Big) \leq e^{-\beta_0\varphi}\ g_0(e),$$

and because g_0 is positive, for $\alpha > \alpha_0$,

$$E_{(e,\varphi)}\Big(e^{-\alpha H_y}\ g_0(X_{H_y})\Big) \leq E_{(e,\varphi)}\Big(e^{-\alpha_0 H_y}\ g_0(X_{H_y})\Big) \leq e^{-\beta_0(\varphi-y)}\ g_0(e). \tag{3.11}$$

By Lemma 3.3, for $\varphi = 0$ and $y > 0$,

$$e^{-\beta_0 y}\ g_0(e) \geq E_{(e,0)}\Big(e^{-\alpha H_y}\ g_0(X_{H_y})\Big) = \begin{pmatrix} e^{yG_\alpha^+}g_0^+ \\ \Pi_\alpha^+ e^{yG_\alpha^+}g_0^+ \end{pmatrix},$$

which implies that $e^{y(G_\alpha^+-\beta_0)}g_0^+ \leq g_0^+$. Hence, because

$$\lim_{y\to 0}\frac{e^{y(G_\alpha^+-\beta_0)}g_0^+ - g_0^+}{y} = (G_\alpha^+-\beta_0)g_0^+,$$

$(G_\alpha^+-\beta_0)g_0^+ \leq 0$ and therefore, because $(G_0^+)^{-1}$ is positive,

$$(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+ 1^+ = (G_0^+)^{-1}(G_\alpha^+-\beta_0 I)g_0^+ \leq 0$$

and $(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+$ is a Q-matrix. It can be proved in the same way that $(G_0^-)^{-1}(G_\alpha-\beta_0 I)G_0^-$ is a Q-matrix. □

Theorem 3.1 *For $\alpha \geq 0$, Let $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ be the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ with maximal negative and minimal positive real parts, respectively, and let $f_{max}(\alpha)$ and $g_{min}(\alpha)$ be their associated eigenvectors, respectively.*

Then, in the oscillating case, there exists $\varepsilon > 0$ such that, for $0 < \alpha < \varepsilon$, and some constants d_n, $n = 2, 3, \ldots$ and $c > 0$,

$$\alpha_{max}(\alpha) = -\frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + d_2\alpha + d_3\alpha^{\frac{3}{2}} + \ldots = -\frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + \Theta_{max}(\alpha^{\frac{1}{2}})$$
$$\beta_{min}(\alpha) = \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + d_2\alpha - d_3\alpha^{\frac{3}{2}} + \ldots = \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + \Theta_{min}(\alpha^{\frac{1}{2}}),$$

and $|\Theta_{max}(\alpha^{\frac{1}{2}})| < c\ \alpha$ and $|\Theta_{min}(\alpha^{\frac{1}{2}})| < c\ \alpha$.

The vectors $f_{max}(\alpha)$ and $g_{min}(\alpha)$ can be chosen to be

$$f_{max}(\alpha) = 1 - \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \alpha\ v_2 + \ldots = 1 - \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \Xi_{max}(\alpha^{\frac{1}{2}})$$
$$g_{min}(\alpha) = 1 + \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \alpha\ w_2 + \ldots = 1 + \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \Xi_{min}(\alpha^{\frac{1}{2}}),$$

where $V^{-1}Qr = 1$, and $|\Xi_{max}(\alpha^{\frac{1}{2}})| < \alpha\ v$ and $|\Xi_{min}(\alpha^{\frac{1}{2}})| < \alpha\ w$ for some positive vectors v and w on E.

In the negative drift case, there exists $\varepsilon > 0$ such that, for $0 < \alpha < \varepsilon$ and some constants a_n and b_n, $n \in \mathbb{N}$,

$$\alpha_{max}(\alpha) = \alpha_{max} + a_1\alpha + a_2\alpha^2 + \ldots \quad \text{and} \quad \beta_{min}(\alpha) = b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \ldots,$$

and the vectors $f_{max}(\alpha)$ and $g_{min}(\alpha)$ can be chosen to be

$$f_{max}(\alpha) = f_{max} + \alpha v_1 + \alpha^2 v_2 + \ldots \quad \text{and} \quad g_{min}(\alpha) = 1 + \alpha w_1 + \alpha^2 w_2 + \ldots,$$

where v_n and w_n, $n \in \mathbb{N}$, are some constant vectors.

The analogous result follows in the positive drift case.

Proof: The eigenvalues of $V^{-1}(Q - \alpha I)$ converge to the eigenvalues of $V^{-1}Q$ as $\alpha \to 0$. Thus, $\alpha_{max}(\alpha) \to \alpha_{max}$ and $\beta_{min}(\alpha) \to \beta_{min}$ as $\alpha \to 0$.

In the drift cases, by (3.8), $\alpha_{max} \neq \beta_{min}$. Hence, α_{max} and β_{min} are simple eigenvalues of $V^{-1}Q$ which implies that, for sufficiently small $\alpha > 0$, $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$, and also $f_{max}(\alpha)$ and $g_{min}(\alpha)$, can be represented by convergent power series (see Wilkinson (1965)). In addition, in the positive drift case, $\alpha_{max} = 0$ and $f_{max} = 1$ and in the negative drift case $\beta_{min} = 0$ and $g_{min} = 1$. Therefore, the part of the theorem for the drift cases is proved.

In the oscillating case, by (3.8), zero is an eigenvalue of the matrix $V^{-1}Q$ with algebraic multiplicity two. Hence, there exists $\varepsilon > 0$ such that for $0 < |\alpha| < \varepsilon$ there exist two eigenvalues of $V^{-1}(Q - \alpha I)$ which converge to zero as $\alpha \to 0$, and those are $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$. In addition, one of the following is valid:

either

$$\begin{aligned} \alpha_{max}(\alpha) &= a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots \\ \beta_{min}(\alpha) &= b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \dots, \end{aligned} \tag{3.12}$$

for some constants a_k, b_k, $k \in \mathbb{N}$, or

$$\begin{aligned} \alpha_{max}(\alpha) &= d_1\alpha^{\frac{1}{2}} + d_2\alpha + d_3\alpha^{\frac{3}{2}} + \dots \\ \beta_{min}(\alpha) &= -d_1\alpha^{\frac{1}{2}} + d_2\alpha - d_3\alpha^{\frac{3}{2}} + \dots, \end{aligned} \tag{3.13}$$

for some constants d_k, $k \in \mathbb{N}$. We shall show that (3.12) is not possible.

For any $\alpha > 0$,

$$(Q - \alpha_{max}(\alpha)V)f_{max}(\alpha) = \alpha f_{max}(\alpha). \tag{3.14}$$

Since by Lemma 3.1, the matrix $(Q - \alpha_{max}V)$ is irreducible and essentially non-negative and the vector $f_{max}(\alpha)$ is positive, it follows that α is the Perron-Frobenius eigenvalue of $(Q - \alpha_{max}(\alpha)V)$. Similarly, α is the Perron-Frobenius eigenvalue of $(Q - \beta_{min}(\alpha)V)$.

Let $\beta \in \mathbb{R}$ and consider the matrix $(Q - \beta V)$ and its Perron-Frobenius eigenvalue $\alpha(\beta)$ and eigenvector $u(\beta)$. The eigenvalue $\alpha(\beta)$ is simple and it converges to a simple eigenvalue of the matrix Q as $\beta \to 0$. Thus, for $|\beta| < \delta$,

$$\begin{aligned} \alpha(\beta) &= c_0 + c_1\beta + c_2\beta^2 + \dots \\ u(\beta) &= 1 + \beta v_1 + \beta^2 v_2 + \dots, \end{aligned} \tag{3.15}$$

for some constants c_k, $k \in \mathbb{N} \cup \{0\}$ and some vectors v_k, $k \in \mathbb{N}$, on E.

Suppose that the process $(\varphi_t)_{t\geq 0}$ oscillates. By Lemmas 3.9 and 3.10 the eigenvalue $\alpha(\beta)$ attains its global minimum 0 at $\beta = 0$. Hence, $\alpha(0) = \frac{d\alpha}{d\beta}(0) = 0$, which gives that $c_0 = c_1 = 0$, and therefore

$$\alpha(\beta) = c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + \dots\ . \tag{3.16}$$

By substituting $\alpha(\beta)$ and $u(\beta)$ into the equation

$$(Q - \beta V)u(\beta) = \alpha(\beta)u(\beta)$$

and by equating terms in β and β^2 on each side of the previous equation, we obtain

$$V^{-1}Qv_1 = 1 \qquad Qv_2 - Vv_1 = c_2 1. \tag{3.17}$$

It follows that $c_2 \neq 0$ (if $c_2 = 0$ then $V^{-1}Qv_2 = v_1$ which is by Jordan matrix theory not possible since 0 is the eigenvalue of $V^{-1}Q$ with algebraic multiplicity 2).

Suppose that (3.12) is true. Then, it follows from (3.12) and (3.14) that, for $|\alpha| < \varepsilon$,

$$\begin{aligned}\alpha = \alpha(\alpha_{max}) &= c_2\ \alpha_{max}^2(\alpha) + c_3\ \alpha_{max}^3(\alpha) + \ldots \\ &= c_2(a_1\alpha + a_2\alpha^2 + \ldots)^2 + c_3(a_1\alpha + a_2\alpha^2 + \ldots)^3 + \ldots \\ &= c_2 a_1^2\alpha^2 + const.\alpha^3 + \ldots,\end{aligned}$$

which is not possible for every $|\alpha| < \varepsilon$. Hence, (3.12) is not true and thus (3.13) holds.

Substituting $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ from (3.13) into (3.16) gives $d_1^2 = \frac{1}{c_2}$. By Lemmas 3.9 and 3.10, $\alpha(0) = 0$ is the minimum of the function $\alpha(\beta)$ which implies that $\alpha(\beta) > 0$ for all $\beta \in \mathbb{R}$, and, by (3.16), that $c_2 > 0$. By multiplying second equality in (3.17) by μ from the left, we obtain (because $\mu 1 = 1$), $c_2 = \frac{-\mu V v_1}{\mu 1} = -\mu V v_1$. Therefore, the statement in the theorem follows from (3.13) and (3.15). □

4 The oscillating case: Proof of Theorem 2.1

We start by looking at $\lim_{T\to+\infty} P_{(e,\varphi)}^{(T)}(A)$ for $A \in \mathcal{F}_t$. By Lemma 3.5 (viii), the events $\{H_0 > T\}$, $T > 0$, are of positive probability. Thus, for $0 < t < T$ and $A \in \mathcal{F}_t$,

$$P_{(e,\varphi)}^{(T)}(A) = P_{(e,\varphi)}(A \mid H_0 > T) = \frac{E_{(e,\varphi)}\Big(I(A) P_{(X_t,\varphi_t)}(H_0 > T - t) I\{H_0 > t\}\Big)}{P_{(e,\varphi)}(H_0 > T)}. \quad (4.18)$$

First we show that $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T-t)}{P_{(e,\varphi)}(H_0>T)}$ exists by looking at the asymptotic behaviour of the function $t \mapsto P_{(e,\varphi)}(H_0 > t)$.

In the oscillating case, by (3.8) and Lemma 3.5 (iv), zero is an eigenvalue of $V^{-1}Q$ with algebraic multiplicity two and geometric multiplicity one. Therefore, there exists a vector r such that $V^{-1}Qr = 1$. Since the choice of such vector is not relevant in the presented work, we shall always refer to it as if it was fixed.

Let μ be the invariant measure of the process $(X_t)_{t\geq 0}$.

Lemma 4.1 *For any* $(e,\varphi) \in E_0^+$,

$$(i)\quad P_{(e,\varphi)}(H_0 > t) \sim \frac{1}{\pi}\frac{1}{\sqrt{-\mu V r}}\, t^{-\frac{1}{2}}\, (-e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e)), \qquad t \to +\infty,$$

$$(ii)\quad h_r(e,\varphi) \equiv -e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r > 0.$$

Proof: (i) The statement is proved by applying Tauberian theorems to the Laplace transform $\frac{1-E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha}$ of $P_{(e,\varphi)}(H_0 > t)$. By Lemmas 3.2 and 3.3, for $\alpha > 0$ and $(e,\varphi) \in E_0^+$,

$$\frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} = e^{-\varphi V^{-1}Q} \frac{1 - \Gamma_\alpha J_2 1}{\alpha}(e) - \frac{e^{-\varphi V^{-1}(Q-\alpha I)} - e^{-\varphi V^{-1}Q}}{\alpha}\Gamma_\alpha J_2 1(e). \quad (4.19)$$

Let $\beta_{min}(\alpha)$ be the eigenvalue of $V^{-1}(Q - \alpha I)$ with minimal positive real part and let $g_{min}(\alpha)$ be its associated eigenvector. Then, by (3.4), $\Pi^-_\alpha g^-_{min}(\alpha) = g^+_{min}(\alpha)$ and by substituting $g_{min}(\alpha)$ from Theorem 3.1 we obtain, for sufficiently small α

$$\begin{aligned}\frac{1^+ - \Pi^-_\alpha 1^-}{\alpha} &= -\frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, (r^+ - \Pi^- r^-) + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}} (\Pi^-_\alpha - \Pi^-) r^- \\ &\quad + \frac{1}{\alpha}\, \Xi^+_{min}(\alpha^{\frac{1}{2}}) + \frac{1}{\alpha}\, \Pi^-_\alpha\, \Xi^-_{min}(\alpha^{\frac{1}{2}}). \end{aligned} \tag{4.20}$$

By Theorem 3.1, $\frac{1}{\alpha}\, \Xi^+_{min}(\alpha^{\frac{1}{2}})$ is bounded, and by Lemma 3.5 (v), $\Pi^-_\alpha - \Pi^- \to 0$ as $\alpha \to 0$. Thus, it follows from (4.20) that

$$\frac{1^+ - \Pi^-_\alpha 1^-}{\alpha} \sim -\frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, (r^+ - \Pi^- r^-), \qquad \alpha \to 0. \tag{4.21}$$

Since

$$\frac{1 - \Gamma_\alpha J_2 1}{\alpha} = \begin{pmatrix} \frac{1^+ - \Pi^-_\alpha 1^-}{\alpha} \\ 0 \end{pmatrix} \quad \text{and} \quad J_1 \Gamma_2 r = \begin{pmatrix} r^+ - \Pi^- r^- \\ 0 \end{pmatrix},$$

it follows that

$$e^{-\varphi V^{-1} Q} \frac{1 - \Gamma_\alpha J_2 1}{\alpha} \sim -\frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, e^{-\varphi V^{-1} Q} J_1 \Gamma_2 r, \quad \alpha \to 0.$$

The function $\alpha \mapsto e^{-\varphi V^{-1}(Q - \alpha I)}$ is analytic for all α and by Lemma 3.5 (v), $\Gamma_\alpha \to \Gamma$, as $\alpha \to 0$. Hence, the second term on the right-hand side of (4.19) is bounded for small $\alpha > 0$. Therefore, for any $(e, \varphi) \in E \times (0, +\infty)$,

$$\frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} \sim -\frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, e^{-\varphi V^{-1} Q} J_1 \Gamma_2 r(e), \qquad \alpha \to 0.$$

The assertion in the lemma now follows from the Tauberian theorem (see Feller (1971) part 2, XIII.5),

(ii) We give only the sketch of the proof. For the details see Najdanovic (2003) or refer to Jacka et al. (2005).

For any $y \in \mathbb{R}$, let the matrices A_y and C_y be the components of the matrix $e^{-yV^{-1}Q}$ given by

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}.$$

Then, for any $\varphi \in \mathbb{R}$,

$$e^{-\varphi V^{-1} Q} J_1 \Gamma_2 r = \begin{pmatrix} A_\varphi (r^+ - \Pi^- r^-) \\ C_\varphi (r^+ - \Pi^- r^-) \end{pmatrix}.$$

The proof of the lemma consists of showing first that the vector $A_\varphi(r^+ - \Pi^- r^-)$ has a constant sign and that that the vector $C_\varphi(r^+ - \Pi^- r^-)$ has the same constant sign, which implies that the function h_r has a constant sign. Then we deduce from (i) that h_r must be negative.

By ordinary matrix algebra and equalities $e^{yV^{-1}Q}r = r + y1$ and $\Pi^- 1^- = 1^+$, it can be shown that, for any $\varphi, y \in \mathbb{R}$, the vector $A_\varphi(r^+ - \Pi^- r^-)$ satisfies the equality

$$\left(A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right) A_\varphi(r^+ - \Pi^- r^-) = A_\varphi(r^+ - \Pi^- r^-).$$

In addition, it can be shown that the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$, $\varphi \neq y$, is positive and that its Perron-Frobenius eigenvalue is 1. Then, the last equality implies that the vector $A_\varphi(r^+ - \Pi^- r^-)$ is its Perron-Frobenius eigenvector, and therefore has a constant sign.

Furthermore, it can be shown that the matrix $C_\varphi A_\varphi^{-1}$ is positive. Hence, because $C_\varphi(r^+ - \Pi^- r^-) = C_\varphi A_\varphi^{-1}\ A_\varphi(r^+ - \Pi^- r^-)$ and because $A_\varphi(r^+ - \Pi^- r^-)$ has a constant sign, we deduce that the vector $C_\varphi(r^+ - \Pi^- r^-)$ has the same constant sign. Thus, the function h_r has a constant sign, and since $P_{(e,\varphi)}(H_0 > t) > 0$, it follows from (i) that the function h_r is negative. □

For the proof of Theorem 2.1 we need two more lemmas.

Lemma 4.2 *(i) Let $\{f_n, n \in \mathbb{N}\}$ and f be non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $Ef_n = Ef = 1$, where expectation is taken with respect to the probability measure P. If $f_n \to f$ a.s. as $n \to +\infty$, then $f_n \to f$ in $L^1(\Omega, \mathcal{F}, P)$ as $n \to +\infty$.*
(ii) Let $\{P_n, n \in \mathbb{N}\}$ and P be probability measures on a measurable space $(\Omega, \mathcal{F})$ such that, for any $A \in \mathcal{F}$, $P_n(A) \to P(A)$ as $n \to +\infty$. Then the measures $\{P_n, n \in \mathbb{N}\}$ converge weakly to P on $\mathcal{F}$.

Proof: (i) Since $\{f_n, n \in \mathbb{N}\}$ and f are non-negative and $Ef_n = Ef = 1$, the functions $\{f_n(\omega), n \in \mathbb{N}\}$ and $f(\omega)$, $\omega \in \Omega$, are densities with respect to the measure P. In addition, $f_n \to f$ a.s. as $n \to +\infty$ and so $f_n \to f$ in probability as $n \to +\infty$. Therefore, by Theorem 2.2. from Jacka, Roberts (1997), $f_n \to f$ in $L^1(\Omega, \mathcal{F}, P)$ as $n \to +\infty$.
(ii) Let for any $A \in \mathcal{F}$, $P_n(A) \to P(A)$ as $n \to +\infty$. Then, by the definition of strong convergence in Jacka *et.al* (1997), the measures $\{P_n, n \in \mathbb{N}\}$ converge strongly to P which, by Theorem 2.1. in Jacka *et.al* (1997), implies that the measures $\{P_n, n \in \mathbb{N}\}$ converge weakly to P. □

Lemma 4.3 *The function $h_r(e, \varphi)$ is harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and the process $\{h_r(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$.*

Proof: The function h_r is continuously differentiable in φ which by (3.6) implies that h_r is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\geq 0}$ and that $\mathcal{G}h_r = 0$. Hence, the function $h_r(e,\varphi)$ is harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and the process $(h_r(X_t,\varphi_t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$. It follows that the process $(h_r(X_{t\wedge H_0},\varphi_{t\wedge H_0}) = h_r(X_t,\varphi_t)I\{t < H_0\})_{t\geq 0}$ is also a local martingale under $P_{(e,\varphi)}$ (the equality of the processes is valid because $h_r(X_{H_0},\varphi_{H_0}) = 0$ if the process $(X_t,\varphi_t)_{t\geq 0}$ starts in E_0^+). Since the process $\{h_r(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is bounded on every finite interval, it follows that it is a martingale under $P_{(e,\varphi)}$. □

Proof of Theorem 2.1: By Lemmas 4.1 (ii) and 4.3, the function $h_r(e,\varphi)$ is positive and harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$. Therefore, the measure $P^{h_r}_{(e,\varphi)}$ is well-defined.

For fixed $(e,\varphi) \in E_0^+$ and $t \geq 0$, and any $T \geq 0$, let Z_T be a random variable defined by

$$Z_T = \frac{P_{(X_t,\varphi_t)}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)}\; I\{t < H_0\}.$$

Then, by Lemmas 4.1, 4.2 and 4.3 the random variables Z_T converge in $L^1(\Omega,\mathcal{F},P_{(e,\varphi)})$ as $T \to +\infty$ to the random variable $\frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}$. Therefore, by (4.18), for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\begin{aligned}\lim_{T\to+\infty} P^{(T)}_{(e,\varphi)}(A) &= \lim_{T\to+\infty} E_{(e,\varphi)}\Big(I(A)\; Z_T\Big)\\ &= E_{(e,\varphi)}\Big(I(A)\frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}\Big) = P^{h_r}_{(e,\varphi)}(A),\end{aligned}$$

which, by Lemma 4.2 (ii), implies that the measures $(P^{(T)}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to $P^{h_r}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $T \to \infty$. □

5 The negative drift case: Proof of Theorem 2.2

We start again by looking at $\lim_{T\to+\infty} P^{(T)}_{(e,\varphi)}(A)$ for $A \in \mathcal{F}_t$. As in the oscillating case, we need to find $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$.

We recall that β_0 denotes the point at which the Perron-Frobenius eigenvalue $\alpha(\beta)$ of the matrix $(Q - \beta V)$ attains its global minimum (see Lemma 3.9), that $\alpha_0 = \alpha(\beta_0)$ and g_0 denote the Perron-Frobenius eigenvalue and right eigenvector, respectively, of the matrix $(Q - \beta_0 V)$ and that G_0 denotes the diagonal matrix $diag(g_0(e))$. We recall the $E \times E$ matrix Q^0 is given by (2.1) as

$$Q^0(e,e') = G_0^{-1}(Q - \alpha_0 I - \beta_0 V)G_0(e,e').$$

By Lemma 3.11 the matrix Q^0 is a conservative irreducible Q-matrix. Let $(V^{-1}Q)\Gamma^0 = \Gamma^0 G^0$ be the unique Wiener-Hopf factorization of the matrix $V^{-1}Q^0$ and let $\Gamma_2^0 = J\Gamma^0 J$.

Our aim is to prove

Lemma 5.1

(*i*) $h_{r^0}(e,\varphi,t) \equiv -e^{-\alpha_0 t}\, e^{-\beta_0\varphi}\, G_0 e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e) > 0,\ (e,\varphi,t) \in E_0^+ \times [0,+\infty),$

(*ii*) *if* $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T-t)}{P_{(e,\varphi)}(H_0>T)}$ *exists it is equal to* $\frac{h_{r^0}(e',\varphi',t)}{h_{r^0}(e,\varphi,0)}$.

For the proof of the lemma we will need some auxiliary lemmas. For $\alpha > 0$ let $V^{-1}(Q^0 - \alpha I)\Gamma_\alpha^0 = \Gamma_\alpha^0 G_\alpha^0$ be the unique Wiener-Hopf factorisation of the matrix $V^{-1}(Q^0 - \alpha I)$ and for fixed $(e,\varphi) \in E \times \mathbb{R}$, let a function $L_{(e,\varphi)}(\alpha)$, $\alpha \geq \alpha_0$, be defined by

$$L_{(e,\varphi)}(\alpha) = \frac{1 - e^{-\beta_0\varphi}\, G_0\, e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0)I)}\, \Gamma_{\alpha-\alpha_0}^0\, G_0^{-1} J_2 1}{\alpha}(e). \tag{5.22}$$

By Lemmas 3.3 and 3.12, for $\alpha > 0$,

$$L_{(e,\varphi)}(\alpha) = \frac{1 - e^{-\varphi V^{-1}(Q-\alpha I)}\Gamma_\alpha J_2 1}{\alpha} = \frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} = \int_0^\infty e^{-\alpha t} P_{(e,\varphi)}(H_0 > t)\, dt. \tag{5.23}$$

Lemma 5.2 *For any* $(e,\varphi) \in E_0^+$*, the function* $L_{(e,\varphi)}(\alpha)$ *is analytic for* $Re(\alpha) > \alpha_0$.

Proof: By the definition in Lemma 3.3, the matrices Π_α^+ and Π_α^- are analytic for $Re(\alpha) > 0$. Hence, the matrix Γ_α is analytic for $Re(\alpha) > 0$ and therefore, by Lemma 3.12 the matrix $\Gamma_{\alpha-\alpha_0}^0$ is analytic for $Re(\alpha) > \alpha_0$. It follows that the numerator of $L_{(e,\varphi)}(\alpha)$ in (5.22) is analytic for $Re(\alpha) > \alpha_0$ and since

$$e^{-\beta_0\varphi} G_0 e^{-\varphi V^{-1}(Q^0+\alpha_0 I)} \Gamma_{-\alpha_0}^0 G_0^{-1} J_2 1 = e^{-\varphi V^{-1}Q}\Gamma J_2 1 = 1,$$

the numerator of $L_{(e,\varphi)}(\alpha)$ is equal to zero for $\alpha = 0$. Therefore, $L_{(e,\varphi)}(\alpha)$ is analytic for $Re(\alpha) > \alpha_0$. □

We note that the objects (e.g. vectors and matrices) with the superscript ^{0} are associated with the matrix Q^0 and are defined in the same way as their counterparts associated with the matrix Q.

Lemma 5.3 *Let all non-zero eigenvalues of the matrix* $V^{-1}Q^0$ *be simple. Then, for some non-zero constant* c,

(*i*) $(\Gamma_{\alpha-\alpha_0}^0 - \Gamma^0)\ G_0^{-1} J_2 1 \ \sim\ c\,(\alpha-\alpha_0)^{\frac{1}{2}}\ J_1 \Gamma_2^0 r^0, \quad \alpha \to \alpha_0,$

(*ii*) $L_{(e,\varphi)}(\alpha) - L_{(e,\varphi)}(\alpha_0) \ \sim\ c\,(\alpha-\alpha_0)^{\frac{1}{2}}\ e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e), \quad \alpha \to \alpha_0.$

Proof: (i) Let g^- be a non-negative vector on E^-. Then

$$g^- = \sum_{k=1}^{m} a_k \, g_k^{0,-}$$

for some constants a_k, $k = 1, \dots m$, where the vectors $g_k^{0,-}$, $k = 1, \dots m$, form a basis in the space of all vectors on E^- and are associated with the eigenvalues of the matrix $G^{0,-}$. By Lemma 3.6 (ii), the constant a_{min} which corresponds to $g_{min}^{0,-} = 1^-$ in the previous linear combination is not zero. Thus,

$$\Pi_\alpha^{0,-} g^- = a_{min} \, \Pi_\alpha^{0,-} 1^- + \sum_{g_k^{0,-} \neq g_{min}^{0,-}} a_k \, \Pi_\alpha^{0,-} g_k^{0,-}. \tag{5.24}$$

By (3.8) and Lemma 3.11, the matrices Q^0 and V define the oscillating case. Therefore, by (4.21),

$$1^+ - \Pi_\alpha^{0,-} 1^- \sim -\frac{1}{\sqrt{-\mu^0 V r^0}} \, \alpha^{\frac{1}{2}} \, (r^{0,+} - \Pi^{0,-} r^{0,-}), \qquad \alpha \to 0. \tag{5.25}$$

We also need the behaviour of $\Pi_\alpha^{0,-} g_k^{0,-}$, $k = 1, \dots m$, $g_k^{0,-} \neq g_{min}^{0,-}$. Since by assumption all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, it can be shown (see Wilkinson (1965)) that there exist vectors $v_{k,n}$, $n \in \mathbb{N}$, on E such that

$$\Pi_\alpha^{0,-} g_k^{0,-} - \Pi^{0,-} g_k^{0,-} = \sum_{n=1}^{\infty} \alpha^n \, (v_{k,n}^+ - \Pi_\alpha^{0,-} v_{k,n}^-) \tag{5.26}$$

From (5.24), (5.25) and (5.26), and because by Lemma 3.5 (v), $\Pi_\alpha^{0,-} \to \Pi^{0,-}$ as $\alpha \to 0$,

$$\Pi_\alpha^{0,-} g^- - \Pi^{0,-} g^- \sim -\frac{a_{min}}{\sqrt{-\mu^0 V r^0}} \, \alpha^{\frac{1}{2}} \, (r^{0,+} - \Pi^{0,-} r^{0,-}), \qquad \alpha \to 0,$$

which proves (i).

(ii) By the definition of $L_{(e,\varphi)}(\alpha)$,

$$\begin{aligned} & L_{(e,\varphi)}(\alpha) - L_{(e,\varphi)}(\alpha_0) \\ = \; & -\frac{(\alpha - \alpha_0)(1 - e^{-\beta_0 \varphi} G_0 \, e^{-\varphi V^{-1} Q^0} \Gamma^0 G_0^{-1} J_2 1)}{\alpha \, \alpha_0} \\ & - \frac{\alpha_0 (e^{-\beta_0 \varphi} G_0 \, (e^{-\varphi V^{-1}(Q^0 - (\alpha - \alpha_0))} - e^{-\varphi V^{-1} Q^0}) \Gamma^0 G_0^{-1} J_2 1)}{\alpha \, \alpha_0} \\ & - \frac{e^{-\beta_0 \varphi} G_0 \, e^{-\varphi V^{-1} Q^0} (\Gamma^0_{\alpha - \alpha_0} - \Gamma^0) G_0^{-1} J_2 1}{\alpha} \\ & - \frac{e^{-\beta_0 \varphi} G_0 \, (e^{-\varphi V^{-1}(Q^0 - (\alpha - \alpha_0))} - e^{-\varphi V^{-1} Q^0})(\Gamma^0_{\alpha - \alpha_0} - \Gamma^0) G_0^{-1} J_2 1}{\alpha}. \end{aligned}$$

The function $\alpha \mapsto e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))}$ is analytic for all α which implies that $e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))} - e^{-\varphi V^{-1}Q^0}$ tends to zero as $\alpha \to \alpha_0$. Hence, by (i) and the last equality, (ii) is valid. □

Lemma 5.4 *For fixed $(e,\varphi) \in E_0^+$, the function $L_{(e,\varphi)}(\alpha+\alpha_0)$, $\alpha > 0$, is the Laplace transform of $e^{-\alpha_0 t}P_{(e,\varphi)}(H_0 > t)$.*

Proof: By (5.23) $L_{(e,\varphi)}(\alpha)$, $\alpha > 0$, is a Laplace transform and therefore, by Theorem 1a in Feller (1971) part 2, XIII.4, completely monotone for $\alpha \geq 0$. In addition, by Lemma 5.2, $L_{(e,\varphi)}(\alpha)$ is analytic for $\alpha > \alpha_0$. Since the analytic continuation of a completely monotone function is completely monotone, it follows that $L_{(e,\varphi)}(\alpha)$ is completely monotone for $\alpha > \alpha_0$ and therefore it is a Laplace transform of some measure on $[0,+\infty)$. By the uniqueness of the inverse of the Laplace transform it follows from (5.23) that $L_{(e,\varphi)}(\alpha+\alpha_0)$ for $\alpha > 0$ is the Laplace transform of $e^{-\alpha_0 t}P_{(e,\varphi)}(H_0 > t)$. □

Proof of Lemma 5.1: (i) By Lemma 4.1 (i), the vector $-e^{-\varphi V^{-1}Q^0}J_1\Gamma_2^0 r^0$ is positive for any $\varphi \in \mathbb{R}$. Since the matrix G_0 is positive by its definition, it follows that the function $h_{r^0}(e,\varphi,t)$ is positive for any $(e,\varphi,t) \in E_0^+ \times [0,+\infty)$.

(ii) By Lemma 5.4, $\frac{L_{(e,\varphi)}(\alpha+\alpha_0)}{\alpha} - \frac{L_{(e,\varphi)}(\alpha_0)}{\alpha}$ is the Laplace transform of the monotone function

$$t \mapsto \int_0^t e^{-\alpha_0 s}P_{(e,\varphi)}(H_0 > s)\,ds - L_{(e,\varphi)}(\alpha_0).$$

Therefore, by the Tauberian theorem (see Feller (1971) part 2, XIII.5),

$$\int_0^t e^{-\alpha_0 s}P_{(e,\varphi)}(H_0 > s)\,ds - L_{(e,\varphi)}(\alpha_0) \sim \frac{c}{\Gamma(\frac{1}{2})}\,t^{-\frac{1}{2}}e^{-\beta_0\varphi}G_0\,e^{-\varphi V^{-1}Q^0}J_1\Gamma_2^0 r^0(e),$$

as $t \to +\infty$. Then, for fixed $(e,\varphi),(e',\varphi') \in E_0^+$,

$$\lim_{T\to+\infty}\frac{\int_0^{T-t} e^{-\alpha_0 s}P_{(e',\varphi')}(H_0 > s)\,ds - L_{(e',\varphi')}(\alpha_0)}{\int_0^{T} e^{-\alpha_0 s}P_{(e,\varphi)}(H_0 > s)\,ds - L_{(e,\varphi)}(\alpha_0)} = \frac{e^{-\beta_0\varphi'}G_0\,e^{-\varphi' V^{-1}Q^0}J_1\Gamma_2^0 r^0(e')}{e^{-\beta_0\varphi}G_0\,e^{-\varphi V^{-1}Q^0}J_1\Gamma_2^0 r^0(e)}.$$

The statement in the lemma is now proved since, by L'Hôpital's rule,

$$\lim_{T\to+\infty}\frac{\int_0^{T-t} e^{-\alpha_0 s}P_{(e',\varphi')}(H_0 > s)\,ds - L_{(e',\varphi')}(\alpha_0)}{\int_0^{T} e^{-\alpha_0 s}P_{(e,\varphi)}(H_0 > s)\,ds - L_{(e,\varphi)}(\alpha_0)} = e^{\alpha_0 t}\lim_{T\to+\infty}\frac{P_{(e',\varphi')}(H_0 > T-t)}{P_{(e,\varphi)}(H_0 > T)},$$

if the latter limit exists. □

Lemma 5.5 *The function h_{r^0} is space-time harmonic for the process $(X_t,\varphi_t)_{t\geq 0}$ and the process $\{h_{r^0}(X_t,\varphi_t,t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$.*

Proof: The function h_{r^0} is continuously differentiable in φ and t which by (3.7) implies that it is in the domain of the infinitesimal generator $\mathcal{A}$ of the process $(X_t, \varphi_t)_{t\geq 0}$ and that $\mathcal{A}h_{r^0} = 0$. Hence, the function $h_{r^0}(e, \varphi, t)$ is space-time harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$ and the process $(h_{r^0}(X_t, \varphi_t, t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$. It follows that the process $(h_{r^0}(X_{t\wedge H_0}, \varphi_{t\wedge H_0}, t\wedge H_0) = h_{r^0}(X_t, \varphi_t, t)I\{t < H_0\})_{t\geq 0}$ is also a local martingale under $P_{(e,\varphi)}$. Since the process $h_{r^0}(X_t, \varphi_t, t)I\{t < H_0\})_{t\geq 0}$ is bounded on every finite interval, it follows that it is a martingale under $P_{(e,\varphi)}$. □

Proof of Theorem 2.2: By Lemmas 5.1 (i) and 5.5, the function $h_{r^0}(e, \varphi, t)$ is positive and space-time harmonic for the process $(X_t, \varphi_t)_{t\geq 0}$. Therefore, the measure $P^{h_{r^0}}_{(e,\varphi)}$ is well-defined.

For fixed $(e, \varphi) \in E_0^+$ and $t \geq 0$ and any $T \geq 0$, let Z_T be a random variable defined by

$$Z_T = \frac{P_{(X_t,\varphi_t)}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)}\ I\{t < H_0\}.$$

Then, by Lemmas 5.1, 4.2 (i) and 5.5 the random variables Z_T converge to $\frac{h_{r^0}(X_t,\varphi_t)}{h_{r^0}(e,\varphi)} I\{t < H_0\}$ in $L^1(\Omega, \mathcal{F}, P_{(e,\varphi)})$ as $T \to +\infty$. Therefore, by (4.18), for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\lim_{T\to+\infty} P^{(T)}_{(e,\varphi)}(A) = \lim_{T\to+\infty} E_{(e,\varphi)}\Big(I(A)\ Z_T\Big) = P^{h_{r^0}}_{(e,\varphi)}(A),$$

which, by Lemma 4.2 (ii), implies that the measures $(P^{(T)}_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to $P^{h_{r^0}}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $T \to \infty$. □

References

[1] Barlow, M. T, Rogers, L. C. G., Williams, D. (1980). Wiener-Hopf factorization for matrices. Seminaire de Probabilites XIV, Springer Lecture Notes in Math., 784, 324-331.

[2] Bertoin, J. and Doney, R. A. (1994). On conditioning a random walk to stay non-negative. Ann. Probab. Vol. 22, No. 4, 2152-2167.

[3] Cohen, J. C. (1981). Convexity of the dominant eigenvalue of an essentially non-negative matrix. Proc. Amer. Math. Soc. Vol. 81, No. 4, 656-658.

[4] Doetsch, G. (1974). *Introduction to the Theory and Application of the Laplace Transformation.* Berlin Heidelberg New York: Springer-Verlag.

[5] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications.* John Wiley and Sons, Inc.

[6] Iglehart, D. L. (1974). Random walks with negative drift conditioned to stay positive. J. Appl. Prob. 11, 742-751.

[7] Jacka, S. D., Lazic, Z., Warren, J. (2005) Conditioning an additive functional of a Markov chain to stay non-negative II: hitting a high level. Submitted to J. Appl. Prob.

[8] Jacka, S. D. and Roberts, G. O. (1988) Conditional diffusions: their infinitesimal generators and limit laws. Research report, Dept. of Statistics, University of Warwick.

[9] Jacka, S. D. and Roberts, G. O. (1995) Weak convergence of conditioned processes on a countable state space. J. Appl. Prob. 32, 902-916.

[10] Jacka, S. D. and Roberts, G. O. (1997) On strong forms of weak convergence. Stoch. Proc. Their Apl. 67, 41-53.

[11] Jacka, S. D. and Warren, J. (2002) Examples of convergence and non-convergence of Markov chains conditioned not to die. Electron. J. Probab. 7 (2002), No. 1, 22 p.

[12] Knight, F. B. (1969). Brownian local times and taboo processes. Trans. Amer. Math. Soc. (143), 173–185.

[13] London, R. R., McKean, H. P., Rogers, L. C. G., Williams, D. (1980). A martingale approach to some Wiener-Hopf problems II. Seminaire de Probabilites XVI, Springer Lecture Notes in Math., 920, 68-91.

[14] Najdanovic, Z. (2003). Conditioning a Markov chain upon the behavoiur of an additive functional. PhD Thesis, University of Warwick.

[15] Pinsky, R. G. (1985). On the convergence of diffusions processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. Ann. Probab. Vol. 13, No. 2, 363-378.

[16] Seneta, E. (1981). *Non-negative Matrices and Markov Chains*. New York Heidelberg Berlin: Springer-Verlag.

[17] Wilkinson, J. H. (1965). *The Algebraic Eigenvalue Problem*. Clarendon Press Oxford.

Authors:

Saul D. Jacka, Department of Statistics, University of Warwick, Coventry, CV4 7AL, S.D.Jacka@warwick.ac.uk

Zorana Lazic, Department of Mathematics, University of Warwick, Coventry, CV4 7AL, Z.Lazic@warwick.ac.uk

Jon Warren, Department of Statistics, University of Warwick, Coventry, CV4 7AL, J.Warren@warwick.ac.uk