Devolving Command under Conflicting Military Objectives

J.Q.Smith, L.Dodd and J. Moffat
University of Warwick, QinetiQ & Dstl

March 17, 2008

Abstract

Command and Control (C2) arrangements aim to ensure their autonomous but rational commanders act in a way that is consistent with C2's overall objectives. This can be a challenge, especially in scenarios where individual commanders face conflicting objectives: the success of a mission verses the compromise of a campaign. Building on our experiences in observing the reactions of military in simulated conflict, in this paper we demonstrate how multiattribute utility theory can be used to model the effects of such conflicting objectives on a particular commander's actions. We show that the geometrical forms of expected utilities that arise from the assumption of commander rationality are qualitatively stable in a wide range of scenarios and therefore open to analysis. We proceed to demonstrate how an appreciation of this geometry can aid understanding of the relationship between complex operational environments and the C2 arrangements and also inform selection and training of personnel to address such conflicts appropriately.

1 Introduction

It is not uncommon for command and control (C2) arrangements to devolve aspects of its military decision making. For example in the UK and Israel, [9], it has proved efficacious to communicate mission objectives in broad terms only and devolve real time tactical decision making to a commander who is best able to appreciate what is happening on the ground. C2 and its field personnel are therefore players in a collaborative game. Remote agile higher level commanders, knowing only some aspects of the geometry of their autonomous commander's utility functions, need to determine when to devolve decision making and when to communicate more prescriptively. In this paper, building on our observations of the behavior of experienced commanders in simulated battle games, [3], we develop a formal framework within which C2 arrangements can be related via commanders' capabilities to the demands of the operational context. We focus on those scenarios which are most difficult to manage: i.e. those when current mission objectives conflict with broader campaign objectives.

C2 arrangements generally aim to encourage *contiguity*: i.e. encourage commanders of different battle groups in geographic or operational proximity to choose acts that are strategically coherent. For if this does not happen then, for example, one commander could order a halt whilst another, not in direct communication, a hasty attack, with potentially disastrous consequences. C2 arrangements should also strive to avoid facing a rational commander with *contradiction* i.e. to avoid encouraging their commanders to commit to a decision who subsequently wish had been completely different. Such stresses can lead to, for example, hypervigilance [2], [8] and can jeopardize a commander's ability to subsequently act rationally. Furthermore, whilst smooth modifications in intensity of engagement are often possible and can be taken at limited cost, dramatic changes - where the commander faced with contradiction tries to dramatically adjust midstream - can be very costly in a wide range of scenarios.

Commanders are expected to act rationally and take full account of their training and experience. Here we interpret this expectation in a Bayesian way: commanders should choose a course of action maximizing their expected utility. Explicitly we assume that a commander chooses a decisive act $\boldsymbol{d} \in \boldsymbol{D}$ from the potentially infinite set of decisions $\boldsymbol{D}$ available so as to maximize the expectation of her utility function U. However it would not be reasonable for higher command to expect its personnel to try to evaluate and take into account the potential acts of all other contiguous autonomous commanders. So each commander will be treated as if they are an autonomous rational agent of C2.

The simplest way to capture the conflict scenario described above is to assume that each commander's utility function $U(\boldsymbol{d}, \boldsymbol{x}|\boldsymbol{\lambda}_1)$ has 2 *value independent attributes* (v.i.a.) $\boldsymbol{x} = (x_1, x_2)$, [5] with parameter vector $\boldsymbol{\lambda}_1$. The first attribute measures the ongoing outcome state of the current mission. The second measures the extent the integrity of a campaign is preserved. Under this assumption, for all decisions $\boldsymbol{d} \in \boldsymbol{D}$ and $x_i \in \mathcal{X}_i$ where $\mathcal{X}_i$ is the sample space of the attribute i $(i = 1, 2)$ the commander's utility function has the form

$$U(\mathbf{d}, x|\boldsymbol{\lambda}_1) = k_1(\boldsymbol{\lambda}_1)U_i(\boldsymbol{d}, x_1|\boldsymbol{\lambda}_1) + k_2(\boldsymbol{\lambda}_1)U_2(\boldsymbol{d}, x_2|\boldsymbol{\lambda}_1)$$

where each *marginal utility* $U_i(\boldsymbol{d}, x_i|\boldsymbol{\lambda}_1)$ is a function of its arguments only and the *criteria weights* $k_i(\boldsymbol{\lambda}_1)$ satisfy $k_i(\boldsymbol{\lambda}_1) \geq 0$, $i = 1, 2$, $k_1(\boldsymbol{\lambda}_1) + k_1(\boldsymbol{\lambda}_1) = 1$, [7],[4]. The rational commander then chooses a decision $\boldsymbol{d}^*(\boldsymbol{\lambda}) \in \boldsymbol{D}$ - called a *Bayes decision* - to maximize the expected utility

$$\overline{U}(\boldsymbol{d}|\boldsymbol{\lambda}) = k_1(\boldsymbol{\lambda})\overline{U}_1(\boldsymbol{d}|\boldsymbol{\lambda}) + k_2(\boldsymbol{\lambda})\overline{U}_2(\mathbf{d}|\boldsymbol{\lambda}) \tag{1}$$

where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) \in \Lambda$ - its possible set of values - and

$$\overline{U}_i(\boldsymbol{d}|\boldsymbol{\lambda}) = \int U_i(\boldsymbol{d}, x_i|\boldsymbol{\lambda}_1)p_i(x_i|\boldsymbol{\lambda}_2)dx_i \tag{2}$$

The known vector $\boldsymbol{\lambda}_2$ will be a function of the hyperparameters defining the commander's subjective posterior distribution - here defined by $p_i(x_i|\boldsymbol{\lambda}_2)$ of attribute x_i, $i = 1, 2$. We now investigate the extent to which C2 arrangements can ensure that the commander's marginal utilities and criteria weights appropriately address C2's priorities of retaining contiguity and avoiding - as far as is possible - commander contradiction.

The autonomous rational commander has a free choice of how she chooses the parameters $\boldsymbol{\lambda}_1$. However the observed and appraised commander will have a utility function which will reflect her understanding of the mission and campaign objectives. Qualitatively a commander's courses of action can be classified into three broad categories She can attempt to simultaneously achieve - at least partially - both the mission objective and the broader campaign objectives. Henceforth we will call this type of decision a *compromise*. On the other hand in a scenario where no course of action is likely to attain satisfactory resolution of either the mission or campaign objectives simultaneously, a compromise will be perceived as futile . Rational choice will then need to focus on finding a *combative* action best achieving the mission objective whilst ignoring the broader campaign objectives or alternatively choosing a *circumspect* action - focusing on avoiding jeopardizing the campaign and to essentially abort the mission. The transition from a rational act being a compromise between objectives to a stark choice between combat and circumspection can be explained through examining the geometry of a commander's expected utility function. This geometry is remarkably robust to the choice of parametric models: the type of courses of action being determined by:

1. qualitative features of the descriptors of the conflict,
2. the uncertainty of the mission and
3. the relative importance the commander places on the two objectives.

This robustness allows us to develop a useful general theory for decision making under command conflict and enables us to suggest remedial ways for C2 arrangements to encourage appropriate commander response. In the next section we analyse how the geometry of the corresponding expected utility functions changes qualitatively under different combat scenarios and different types of commander. In section 3 we demonstrate some general properties of rational decision making in this context. In section 4 we discuss how with some mild differentiability conditions our taxonomy relates to the classification of catastrophes [12], [17] and give a number of illustrative examples. We end the paper by relating theory to observed behavior and give some general recommendations for C2 in the light of these geometrical insights.

2 Rational Decisions for Competing Objectives

2.1 A Probabilistic Formulation

The commander's decision space $\boldsymbol{D}$ will typically be very complex, and be constrained by, for example, the available resources and the rules of engagement of the mission. However, for a wide class of scenarios we will be able to express any course of action $\boldsymbol{d} = (d, \boldsymbol{d}_1, \boldsymbol{d}_2) \in \boldsymbol{D} = D \times \boldsymbol{D}_1 \times \boldsymbol{D}_2$ where D is a subset of the real line. In this paper the component d will measure the *intensity* of the engagement in the mission. We assume that increasing the intensity of engagement does not reduce the commander's probability of successfully completing the mission but is believed to have a potentially negative effect on the campaign. Thus it is not unusual for a mission to be successfully addressed by combating the enemy with a large force quickly engaged. However the intensity of the engagement will often increase the potential for casualties both the commander's own unit and to the local civilian population. It is also likely to be increasingly politically dangerous and so be increasingly to the detriment of the campaign objectives.

For a chosen level of intensity d a commander will choose, to the best of her ability, between other courses of action $\boldsymbol{d}_1(d)$ associated with best satisfying the mission objectives given d and between other courses of action $\boldsymbol{d}_2(d)$ associated best preserving the integrity of the campaign. Usually $\boldsymbol{d}_1$ encodes specific tactics involved in achieving the current mission. On the other hand the decision $\boldsymbol{d}_2$ codes the tactics involved in best securing human resources, life and retaining political integrity. Both $\boldsymbol{d}_1(d)$ and $\boldsymbol{d}_2(d)$ will usually be decided by the commander in the field and in response to the developing situation, albeit informed by protocol and training. For the rest of this paper we now assume that it is possible to define the intensity d in such a way that these two subsequent choices do not impinge on one another. Formally this will mean that a commander's expected marginal utility $\overline{U}_i(\boldsymbol{d}|\boldsymbol{\lambda})$ is a function only of $(d, \boldsymbol{d}_i, \boldsymbol{\lambda})$, $(d, \boldsymbol{d}_i) \in D \times \boldsymbol{D}_i, i = 1, 2,$ *where* $D \subseteq \mathbb{R}$.

Now let $\boldsymbol{d}_1^*(d), (\boldsymbol{d}_2^*(d))$ denote respectively a best choice to attain the mission objectives (campaign objectives) for a given intensity d. The assumption above makes it possible to characterize behavior in terms of a one dimensional decision space: see below. Assuming without loss that neither criterion weight is zero, in the appendix we show that by taking a linear transformation of the expression (1), a commander's Bayes decision d^* will maximize the function

$$V(d|\boldsymbol{\lambda}) = e^{\rho(\boldsymbol{\lambda})} P_1(d|\boldsymbol{\lambda}) - P_2(d|\boldsymbol{\lambda}) \tag{3}$$

Here - temporarily suppressing the index $\boldsymbol{\lambda}$, for $i = 1, 2$

$$\begin{aligned} 0 &\leq P_1(d) = (u_1[1] - u_1[0])^{-1} \left(\overline{U}_1(d, \boldsymbol{d}_1^*(d)) - u_1[0]\right) \leq 1 \\ 0 &\leq P_2(d) = (u_2[1] - u_2[0])^{-1} \left(\overline{U}_2(d, \boldsymbol{d}_2^*(d)|\boldsymbol{\lambda}) - u_2[0]\right) \leq 1 \end{aligned}$$

where the *daring* $\rho(\boldsymbol{\lambda})$ satisfies

$$\rho = \rho_1 + \rho_2 \tag{4}$$

where

$$\begin{aligned} \rho_1 &= \log k_1 - \log k_2 \\ \rho_2 &= \log (u_1[1] - u_1[0]) - \log (u_2[1] - u_2[0]) \end{aligned}$$

and where for $i = 1, 2$, $u_i[0] = \inf_{d \in D} \overline{U}_i(d, \boldsymbol{d}_i^*(d))$ and $u_i[1] = \sup_{d \in D} \overline{U}_i(d, \boldsymbol{d}_i^*(d))$ denote the worst and best possible outcome - in the eyes of the commander - for each of the objectives. For technical reasons it will be convenient to reparametrize $\boldsymbol{\lambda}$ so that there is a one to one function from $\boldsymbol{\lambda}$ to $(\rho, \boldsymbol{\lambda}') \in \mathbb{R} \times \Lambda'$. Heuristically $\boldsymbol{\lambda}'$ simply spans the parameters in Λ other than ρ. From the constructions above it is clear that $P_1(d|\boldsymbol{\lambda}), P_2(d|\boldsymbol{\lambda})$ can be chosen so that they are only functions of $\boldsymbol{\lambda}$ through $\boldsymbol{\lambda}'$ and so henceforth will be indexed as $P_1(d|\boldsymbol{\lambda}'), P_2(d|\boldsymbol{\lambda}')$.

Note here that $P_1(d|\boldsymbol{\lambda}')$ $(P_2(d|\boldsymbol{\lambda}'))$ are respectively simply an increasing (decreasing) linear transformations of $\overline{U}_i(d, \boldsymbol{d}_i^*(d)|\boldsymbol{\lambda})$: the commander's expected marginal utility $i = 1, 2$ on taking what she considers to be the best possible decision consistent with choosing an intensity d of engagement. From the definition of d note that the functions $P_i(d|\boldsymbol{\lambda}')$ are each distribution functions in d : i.e. non-decreasing in $d \in D$, with

$$P_i(\inf \{d \in D\} |\boldsymbol{\lambda}') = 0, P_i(\sup \{d \in D\} |\boldsymbol{\lambda}') = 1 \tag{5}$$

parametrized by $\boldsymbol{\lambda}' \in \Lambda'$, and $i = 1, 2$. Denote the smallest closed interval containing the support of $P_i(d|\boldsymbol{\lambda})$ by $[a_i(\boldsymbol{\lambda}'), b_i(\boldsymbol{\lambda}')]$, $i = 1, 2$ where by an abuse of notation we allow any of the lower bounds to take the value $-\infty$ and any of the upper bounds ∞.. Thus a_1 is the value below which the intensity d is useless for attaining any even partial success in the mission. The upper bound b_1 is the lowest intensity that allows the commander to obtain total mission success. Similarly a_2 is the highest value of intensity that can be used without damaging campaign objectives. The bound b_2 is the lowest value at which the campaign is maximally jeopardized. For obvious reasons we will call $b_1(\boldsymbol{\lambda}')$ *pure combat* and $a_2(\boldsymbol{\lambda}')$ *pure circumspection.*.

The meaning of these distributions can be best understood through the following simple but important special case.

Example 1 (zero - one marginal utilities) *When a mission is either fully successful or fails and the campaign is totally compromised or uncompromised then $P_1(d|\boldsymbol{\lambda})$ is the commander's probability that the mission is successful using intensity d and choosing other decisions associated with the mission in the best way she can under this constraint. On the other hand $P_2(d|\boldsymbol{\lambda})$ is the probability that the campaign will be jeopardized if the commander used an intensity d. Note that the difference V defined above, balances these objectives, the relative weight given to mission success being determined by the value of the daring parameter ρ, with equal focus being given when $\rho = 0$.*

In the more common scenarios where the mission can be partially successful the interpretation of $P_i(d|\boldsymbol{\lambda})$, $i = 1, 2$ in fact relates simply to the special case above. Thus, specifically, the partially successful probable consequence of using and intensity d in the given scenario is considered by the commander to be *equivalent* to attaining best possible mission success with probability $P_1(d|\boldsymbol{\lambda})$ and the most jeopardization of the campaign with probability $P_2(d|\boldsymbol{\lambda})$.

One point of interest is that if $V(P_1, P_2, \rho, \boldsymbol{\lambda}')$ is given by (3) and $Q_1 = P_2$, $Q_2 = P_1$ and $\widetilde{\rho} = -\rho$ then $V(Q_1, Q_2, \widetilde{\rho}, \boldsymbol{\lambda}')$ is a strictly decreasing linear transformation of $V(P_1, P_2, \rho, \boldsymbol{\lambda}')$. So in particular these two different settings share the same stationary points but with all local minima of $V(P_1, P_2, \rho, \boldsymbol{\lambda}')$ being local maxima of $V(Q_1, Q_2, \widetilde{\rho}, \boldsymbol{\lambda}')$ and vice versa. Henceforth call $V(Q_1, Q_2, \widetilde{\rho}, \boldsymbol{\lambda}')$ the *dual* of $V(P_1, P_2, \rho, \boldsymbol{\lambda}')$. The close complementary relationship between the geometry of a problem and its dual will be exploited later in the paper.

2.2 Resolvability

Ideally C2's arrangements should be agile, i.e. flexible enough to alternate between devolving decision making to the commander in the field and taking a top-down approach prescribing itself that each commander focus on achieving one or other of the objectives. There are two scenarios where it is straightforward for C2 to decide between devolution and a top down approach. The first occurs when $b_1(\boldsymbol{\lambda}') \leq a_2(\boldsymbol{\lambda}')$. We henceforth call this scenario *resolvable* for $\boldsymbol{\lambda}' \in \Lambda'$ and call the closed interval $\left[b_1(\boldsymbol{\lambda}'), a_2(\boldsymbol{\lambda}')\right]$ the *resolution interval* for $\boldsymbol{\lambda}' \in \Lambda'$. It is easy to see from equation(3) that the set of the commander's optimal decisions require $d^*(\boldsymbol{\lambda}') \in \left[b_1(\boldsymbol{\lambda}'), a_2(\boldsymbol{\lambda}')\right]$ when $V(d^*(\boldsymbol{\lambda}')|\boldsymbol{\lambda}') = \exp \rho(\boldsymbol{\lambda}')$. Note that in particular both pure combat and pure circumspection are always Bayes decisions (as is any level of intensity between). In this case, although the commander's evaluation of her performance $V(d^*(\boldsymbol{\lambda})|\boldsymbol{\lambda})$ is clearly dependent on ρ, her *decision* need not depend on ρ. She simply chooses a moderate intensity of engagement $d^*(\boldsymbol{\lambda}')$ in the interval above enabling the simultaneous optimization on mission and campaign objectives subsequently choosing $d_1^*(\boldsymbol{\lambda}')$ and $d_2^*(\boldsymbol{\lambda}')$ to maximize their respective marginal utility. In fact much military training focuses on this type of scenario, where there exists at least one course of action which is "OK"[9] for both objectives. Good training regimes that ensure the commander can hedge: i.e. identify both $(d^*(\boldsymbol{\lambda}), d_1^*(\boldsymbol{\lambda}))$ and $(d^*(\boldsymbol{\lambda}), d_1^*(\boldsymbol{\lambda}))$ will ensure that a utility maximizing strategy will be found and will not be influenced by the uncertain parameter ρ. C2 should be most prepared to devolve decision making to a commander on the ground when a scenario is resolvable.

A second simple case occurs when $b_2(\boldsymbol{\lambda}') \leq a_1(\boldsymbol{\lambda}')$. Henceforth called this scenario *unresolvable* scenario for $\boldsymbol{\lambda}' \in \Lambda'$. Here there is no possibility of scoring anything from one objective if the commander even partially achieve something in the other. A rational commander's Bayes decision is either pure combat $d^*(\boldsymbol{\lambda}) = b_1(\boldsymbol{\lambda}')$ optimizing mission objectives or pure circumspection $d^*(\boldsymbol{\lambda}) = a_2(\boldsymbol{\lambda}')$ maximizing campaign objectives, choosing the first option iff $\rho \geq 0$. In this scenario C2 therefore knows that a rational commander will apparently ignore completely one or other of the objectives depending on the sign of ρ. It is argued below that ρ can be unpredictable to C2. Therefore in such cases *which* of two extreme reactions will be chosen will be difficult for C2 to predict and control. The agile C2 should therefore be most inclined to be prescriptive in scenarios which are unresolvable and $b_2(\boldsymbol{\lambda}')$ and $a_1(\boldsymbol{\lambda}')$ are far apart enough for the choice between them to cause discontiguity or contradiction.

When scenarios are such that both intervals $[a_i(\boldsymbol{\lambda}'), b_i(\boldsymbol{\lambda}')]$, $i = 1, 2$ are short - i.e. when a commander will judge that the use of an intensity d will either result in complete failure or complete success except in a small range for both the mission or campaign objective - then most scenarios will be resolvable or unresolvable and appropriate C2 arrangements will usually be clear. Of course many scenarios have the property that by using a moderate level of intensity, compromise cannot be expected to fully achieve both objectives - as in the resolvable scenarios - but nevertheless might be a viable possibility - unlike in the unresolvable scenarios. the effect of an intensity d will have intermediate potential success with respect to the mission or campaign over a fairly wide range of values of d. To understand and control the movement from the resolvable to the unresolvable scenario we will henceforth focus on these intermediate scenarios.

Call a scenario a *conflict* when $[a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')]$ is non-empty where $I(\boldsymbol{\lambda}')$ be the open interval defined by

$$I(\boldsymbol{\lambda}') = (a_1(\boldsymbol{\lambda}'), b_1(\boldsymbol{\lambda}')) \cap (a_2(\boldsymbol{\lambda}'), b_2(\boldsymbol{\lambda}')) = (a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}'))$$

The most important scenarios of this type are ones where one of the two intervals in the intersection above is not properly contained in the other. The first - the *primal* conflict scenarios has $a(\boldsymbol{\lambda}') = a_2(\boldsymbol{\lambda}')$ and $b(\boldsymbol{\lambda}') = b_1(\boldsymbol{\lambda}')$. Here the value of intensity at which the campaign begins to become progressively jeopardized is lower than the intensity at which the mission can be ensured to be fully successful. The second case - the *dual* conflict - has $a(\boldsymbol{\lambda}') = a_1(\boldsymbol{\lambda}')$ and $b(\boldsymbol{\lambda}') = b_2(\boldsymbol{\lambda}')$ is more difficult for the commander but has some hope since the intensity required to begin to have some success in the mission is lower than the intensity at which the campaign will be maximally jeopardized.

Note that each primal scenarios with associated potential $V(P_1, P_2, \rho, \boldsymbol{\lambda}')$ with bounds $[a_1, b_1]$ and $[a_2, b_2]$ on P_1, P_2 respectively has a dual scenario associated with its dual $V(Q_1, Q_2, \widetilde{\rho}, \boldsymbol{\lambda}')$ whose bounds on are Q_1, Q_2 are respectively $[a_2, b_2]$ and $[a_1, b_1]$. It follows that the geometry of dual conflicts can be simply deduced from their corresponding primal conflicts. Say a scenario is a *boundary* conflict if $a_1 = a_2$ and $b_1 = b_2$.

Henceforth assume that P_1 and P_2 are absolutely continuous with respective densities p_1 and p_2 and that p_1 and p_2 are strictly positive in the interior of their support and, without loss, zero outside it. Then it is straightforward to check from (3) that when

$$\begin{aligned}
DV(d|\boldsymbol{\lambda}) &= e^{\rho} p_1(d|\boldsymbol{\lambda}') > 0 \text{ when } a_1(\boldsymbol{\lambda}') < a_2(\boldsymbol{\lambda}') \text{ and } d \in (a_1(\boldsymbol{\lambda}'), a_2(\boldsymbol{\lambda}')) \\
DV(d|\boldsymbol{\lambda}) &= -p_2(d|\boldsymbol{\lambda}') < 0 \text{ when } a_1(\boldsymbol{\lambda}') > a_2(\boldsymbol{\lambda}') \text{ and } d \in (a_1(\boldsymbol{\lambda}'), a_2(\boldsymbol{\lambda}')) \\
DV(d|\boldsymbol{\lambda}) &= -p_2(d|\boldsymbol{\lambda}') < 0 \text{ when } b_1(\boldsymbol{\lambda}') < b_2(\boldsymbol{\lambda}') \text{ and } d \in (b_1(\boldsymbol{\lambda}'), b_2(\boldsymbol{\lambda}')) \\
DV(d|\boldsymbol{\lambda}) &= e^{\rho} p_1(d|\boldsymbol{\lambda}') > 0.\text{when } b_1(\boldsymbol{\lambda}') > b_2(\boldsymbol{\lambda}') \text{ and } d \in (b_1(\boldsymbol{\lambda}'), b_2(\boldsymbol{\lambda}'))
\end{aligned}$$

It therefore follows that whatever the value of $\boldsymbol{\lambda}' \in \Lambda'$ we can find a Bayes decision $d^*(\boldsymbol{\lambda}') \in I^+(\boldsymbol{\lambda}')$ where

$$I^+(\boldsymbol{\lambda}') = I(\boldsymbol{\lambda}') \cup \{a_2(\boldsymbol{\lambda}')\} \cup \{b_1(\boldsymbol{\lambda}')\}$$

Henceforth in this paper we will assume, without loss, that the commander chooses her act from $I^+(\boldsymbol{\lambda}')$. So in any conflict scenario $d^*(\boldsymbol{\lambda}') \in [a_2, b_1]$. In a dual scenario $d^*(\boldsymbol{\lambda}')$ is either at the extremes of intensity worth considering a_2 or b_1 or lies in the open interval (a_1, b_2).We next study the effect of the value of the daring ρ on a commander's decisions.

2.3 Daring and intensity of action

Fix the value of $\boldsymbol{\lambda}'$ and suppress this index. Then for each $d > d', d, d' \in I^+(\boldsymbol{\lambda}')$ with the property that $P_2(d) > 0$, there exists a large negative ρ such that

$$V(d|\boldsymbol{\lambda}) - V(d'|\boldsymbol{\lambda}) = e^{\rho} \{P_1(d) - P_1(d')\} - P_2(d') < 0$$

So in this sense as $\rho \to -\infty$ the rational commander will choose a decision increasingly close to pure circumspection a_2. On the other hand for all fixed $\boldsymbol{\lambda}'$ for each $d < d', d, d' \in I^+(\boldsymbol{\lambda}')$ with the property that $P_1(d) > 0$, there exists a large negative ρ such that

$$e^{-\rho} (V(d|\boldsymbol{\lambda}) - V(d'|\boldsymbol{\lambda})) = e^{-\rho} (P_2(d') - P_1(d)) - P_1(d) < 0$$

So as the daring parameter $\rho \to \infty$ becomes large and negative the rational commander will choose a decision increasingly close to pure combat b_1.

Next note that any rational commander will assess that if $d' < d$ and d' is not preferred to d when $\rho = \rho_0$ then d' is not preferred to d when $\rho = \rho_1$ when $\rho_1 \geq \rho_0$. To see this simply note that

$$V(d|\rho_1) - V(d'|\rho_1) = (V(d|\rho_0) - V(d'|\rho_0)) + (e^{\rho_1} - e^{\rho_0}) (P_1(d) - P_1(d'))$$

The first term on the right hand side is non-negative by hypothesis whilst the second is positive since P_1 is a distribution function. Further, by an analogous argument, if $d' > d^*$ and d' is not preferred to d^* when $\rho = \rho_0$ then d' is not preferred to d^* when $\rho = \rho_1$ when $\rho_1 \leq \rho_0$ either. In this sense a rational commander will choose to engage with non-decreasing intensity as ρ increases *whatever* the circumstances. We shall henceforth call this property *$\rho-$ monotonicity*. Let

$$D^*(\rho, \boldsymbol{\lambda}') = \{d^*(\rho, \boldsymbol{\lambda}') : d^*(\rho, \boldsymbol{\lambda}') = \arg\sup V(d^*|\rho, \boldsymbol{\lambda}')\}$$

denote the set of optimal intensities $d^*(\rho, \boldsymbol{\lambda}')$ for a commander whose parameters are $(\rho, \boldsymbol{\lambda}')$. Note that ρ-monotonicity implies that if $D^*(\rho_0, \boldsymbol{\lambda}')$ contains pure circumspection then so does $D^*(\rho, \boldsymbol{\lambda}')$ where $\rho < \rho_0$. Similarly if $D^*(\rho_1, \boldsymbol{\lambda}')$ contains pure aggression then so does $D^*(\rho, \boldsymbol{\lambda}')$ where $\rho > \rho_1$. When for some fixed value $\boldsymbol{\lambda}'$ and for ρ lying in the closed interval $[\rho_0, \rho_1]$, $D^*(\rho, \boldsymbol{\lambda}')$ consists of the single point $\{d^*(\rho, \boldsymbol{\lambda}')\}$. Then the monotonicity condition above and the strict positivity of $p_1(d^*|\boldsymbol{\lambda})$ or $p_2(d^*|\boldsymbol{\lambda})$ on their support then tells us this $d^*(\rho, \boldsymbol{\lambda}') \in I(\boldsymbol{\lambda}')$ is *strictly* increasing $\rho \in [\rho_0, \rho_1]$. So the larger $\rho(\boldsymbol{\lambda}')$ is the higher the priority she places on mission success. From the above this will be reflected in her choice of intensity: the larger the value of $\rho(\boldsymbol{\lambda}')$ the greater her choice of intensity.

Recall form equation(4) that the daring $\rho(\boldsymbol{\lambda}') = \rho_1(\boldsymbol{\lambda}') + \rho_2(\boldsymbol{\lambda}')$ decomposes into two terms. The term $\rho_1(\boldsymbol{\lambda}')$ is an increasing function of the relative weight placed on the mission against the campaign objectives; i.e. her *prioritization*. Note also that it is the only term in V affected by a commander's criterion weights. This term may be potentially very unpredictable to C2, especially if no strong training is given about how to balance mission and broader campaign objectives. Even when this training has happened the personality and emotional history will color the commander's choice of this parameter.

The term $\rho_2(\boldsymbol{\lambda}')$ is an increasing function of how much better the commander believes she can achieve mission over campaign objectives were she able to choose an optimal intensity for either. This, of course depends on the scenario faced and her competence - something that C2 might hope to estimate well. But, since it is based on her *own* evaluation of her competence it also reflects her relative *confidence* in her ability to achieve mission success or be sensitive to the campaign objectives. A commander's lack of training or difficult recent emotional history may well have a big affect on this term. Note that a large positive value of this parameter encourages the commander to focus almost entirely on the mission objectives whilst a large negative value would encourage her to neglect the mission objectives in favour of the overall campaign objectives.

3 The developing bifurcation

3.1 Bifurcation with continuous potentials

Here, building on methodologies developed in [9], [10], [16], we investigate the geometrical conditions determining when bifurcation of the expected utility can occur. When $V(d|\boldsymbol{\lambda})$ is continuous a commander's optimal choice will move smoothly in response to smooth changes in $\boldsymbol{\lambda}$, provided that her best course of action $d^*(\boldsymbol{\lambda})$ is unique: see the appendix for a formal statement of this property and a proof. Thus the undesirable situations of there being dramatic differences between the Bayes decisions of contiguous commanders at $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 = (\rho_0, \boldsymbol{\lambda}_0')$ or a single commander suddenly faced with contradiction can only occur when $D^*(\rho_0, \boldsymbol{\lambda}_0')$ contains at least two Bayes decisions- and hence in particular two local maxima. On the other hand if $D^*(\rho_0, \boldsymbol{\lambda}_0')$ contains two decisions $d_1^*(\rho_0), d_2^*(\rho_0)$ where $d_1^*(\rho_0) < d_2^*(\rho_0)$, then holding $\boldsymbol{\lambda}_0'$ fixed and increasing ρ through ρ_0 from the above we must jump from a $d^*(\rho) \leq d_1^*(\rho_0)$ being optimal $\rho \leq \rho_0$ to a $d^*(\rho) \geq d_2^*(\rho_0)$ being optimal. This in turn implies that C2 can be faced with a lack of contiguity and commander contiguity whenever their daring is near ρ_0. So there is an intimate link between when it is expedient for C2 to delegate and the cardinality of $D^*(\rho_0, \boldsymbol{\lambda}_0')$, which in turn is related to the number of local maxima of $V(d|\boldsymbol{\lambda})$.

Again suppressing the index $\boldsymbol{\lambda}'$ a rational commander will choose a non-extreme option $d^*(\boldsymbol{\lambda}) \in I(\boldsymbol{\lambda}')$ for some value $\rho(\boldsymbol{\lambda}')$ if and only if

$$V(d^*(\boldsymbol{\lambda})|\boldsymbol{\lambda}) = e^{\rho} P_1(d^*(\boldsymbol{\lambda})) - P_2(d^*(\boldsymbol{\lambda})) \geq \max\{e^{\rho} - 1, 0\}$$

i.e.

$$P_1(d^*(\boldsymbol{\lambda})) \left\{P_2(d^*(\boldsymbol{\lambda}))\right\}^{-1} \geq e^{-\rho} \geq \{1 - P_1(d^*(\boldsymbol{\lambda})\} \left\{1 - P_2(d^*(\boldsymbol{\lambda}))\right\}^{-1}$$

or equivalently

$$P_1(d^*(\boldsymbol{\lambda}))\left\{1 - P_1(d^*(\boldsymbol{\lambda}))\right\}^{-1} \geq P_2(d^*(\boldsymbol{\lambda}))\left\{1 - P_2(d^*(\boldsymbol{\lambda}))\right\}^{-1} \tag{6}$$

It follows, in particular, that if for all $d^*(\boldsymbol{\lambda}) \in I(\boldsymbol{\lambda}')$

$$P_1(d^*(\boldsymbol{\lambda})|\boldsymbol{\lambda}') \leq P_2(d^*(\boldsymbol{\lambda})|\boldsymbol{\lambda}') \tag{7}$$

- i.e. P_2 *stochastically dominates* P_1 - then all commanders will have a Bayes decision either pure combat or pure circumspection, their choice depending on their daring, i.e. act just as in an unresolvable scenario. Call such a scenario *pseudo-unresolvable*. Pseudo - unresolvable conflicts have the same difficult consequences as the unresolvable ones for C2 and are therefore strong candidates for prescriptive arrangements. Note that in our zero-one example above a scenario is pseudo-unresolvable iff, for all $d \in I(\boldsymbol{\lambda}')$, the probability of mission success using intensity d is no larger than the probability of jeopardizing the campaign.

When this domination is violated at some point $d_0 \in I(\boldsymbol{\lambda}')$ then C2 will predict that a commander with a particular level of daring will choose an interior decision, so compromise can be a viable option for at least some commanders. At the other extreme when P_1 stochastically dominates P_2 then, for any commander, an interior decision $d^*(\boldsymbol{\lambda}) \in I(\boldsymbol{\lambda}')$ is at least as good as pure combat or circumspection. We now study the position and nature and development of these interior decisions under smoothly changing scenarios and personnel.

3.2 Bifurcation when distributions are twice differentiable

Henceforth assume that the distributions P_i are twice differentiable in the open interval $\left(a_1(\boldsymbol{\lambda}'), b_2(\boldsymbol{\lambda}')\right)$, $i = 1, 2$ and constant nowhere in this interval. On differentiating and taking logs, any local maximum of $V(d|\boldsymbol{\lambda})$ will either lie on the boundary of I or satisfy

$$v(d|\boldsymbol{\lambda}') \triangleq f_2(d|\boldsymbol{\lambda}') - f_1(d|\boldsymbol{\lambda}') = \rho \tag{8}$$

where $f_i(d|\boldsymbol{\lambda}') = \log p_i(d|\boldsymbol{\lambda}')$, $i = 1, 2$ where a necessary condition for this stationary point to be a local maximum of V is that the derivative $Dv(d|\boldsymbol{\lambda}') \geq 0$. So in conflicting scenarios the commander's optimal decision $d^* \in I^+(\boldsymbol{\lambda}')$ will either lie on the boundary of $I(\boldsymbol{\lambda}')$ - as in the unresolvable scenario - or satisfy the equation above.

Let $\xi_1(\boldsymbol{\lambda}')$ $(\xi_1'(\boldsymbol{\lambda}'))$and $\xi_2(\boldsymbol{\lambda}')$ $(\xi_2'(\boldsymbol{\lambda}'))$ respectively denote the mode of $p_1(d|\boldsymbol{\lambda}')$ occurring at the largest (smallest) value of d (and hence the largest(smallest) maximum of $f_1(d|\boldsymbol{\lambda}')$) in $(a_1(\boldsymbol{\lambda}'), b_1(\boldsymbol{\lambda}'))$ and the mode of $p_2(d(\boldsymbol{\lambda})|\boldsymbol{\lambda}') = 0$ occurring at the smallest (largest) value of d (and hence the smallest (largest) maximum of $f_2(d|\boldsymbol{\lambda}')$) in the open interval $(a_2(\boldsymbol{\lambda}'), b_2(\boldsymbol{\lambda}'))$. Note that when P_1 and P_2 are both unimodal $\xi_i(\boldsymbol{\lambda}') = \xi_i'(\boldsymbol{\lambda}')$, $i = 1, 2$. In this case because $\xi_1(\boldsymbol{\lambda}')$ is a point of highest incremental gain in mission we call this point the *mission point* and the intensity $\xi_2(\boldsymbol{\lambda}')$ where the threat to campaign objectives worsens fastest the *campaign point*.

When $\xi_1(\boldsymbol{\lambda}') \leq \xi_2(\boldsymbol{\lambda}')$, for any $d \in [\xi_1(\boldsymbol{\lambda}'), \xi_2(\boldsymbol{\lambda}')]$, $v(d|\boldsymbol{\lambda}')$ is strictly decreasing. It follows that there is at most one solution d^* to (8) for any value of ρ and $Dv(d|\boldsymbol{\lambda}') \geq 0$ so this stationary value $d^* \in \big(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\big)$ is a local maximum of V. Call a (primal) scenario *pseudo-resolvable* if

$$\xi_1(\boldsymbol{\lambda}') \leq a_2(\boldsymbol{\lambda}') \leq b_1(\boldsymbol{\lambda}') \leq \xi_2(\boldsymbol{\lambda}') \tag{9}$$

where a Bayes decision can only occur in the closed interval $[a_2(\boldsymbol{\lambda}'), b_1(\boldsymbol{\lambda}')]$ Clearly in this case for each value of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ there is a unique maximum in this interval moving as a continuous function of $\boldsymbol{\lambda}$.

It follows that C2 should find pseudo-resolvable conflicts almost as desirable as resolvable ones and these are therefore prime candidates for devolved decision making. In particular no rational commander will face the stark combative v. circumspection dichotomy. Furthermore, although her choice of act will depend on ρ, two commanders with similar utility weightings as reflected through their value of ρ will act similarly. So in particular it is rational for them to compromise and if contiguous commanders are matched by their training and emotional history then they will make similar and hence broadly consistent choices. In the particular case when the distributions P_1 and P_2 are unimodal, pseudo-resolvable scenarios occur in primal conflict where the effectiveness of the mission of increasing intensity past $a_2(\boldsymbol{\lambda}')$ is waning up to $b_1(\boldsymbol{\lambda}')$ whilst the effect on mission compromise is accelerating. It therefore makes logical sense for a commander to compromise between these two objectives.

On the other hand when $\xi_2'(\boldsymbol{\lambda}') \leq \xi_1'(\boldsymbol{\lambda}')$ for any $d \in [\xi_2'(\boldsymbol{\lambda}'), \xi_1'(\boldsymbol{\lambda}')]$, $v(d|\boldsymbol{\lambda}')$ is strictly increasing. It follows that there is at most one solution to (8) for any value of ρ and $Dv(d|\boldsymbol{\lambda}') \geq 0$ so this stationary value is a local minimum of V. It is easily checked that a (dual) scenario where

$$\xi_2'(\boldsymbol{\lambda}') \leq a_1(\boldsymbol{\lambda}') \leq b_2(\boldsymbol{\lambda}') \leq \xi_1'(\boldsymbol{\lambda}')$$

is pseudo-unresolvable and a Bayes decision can only be pure combat or pure circumspection.

3.3 Convexity and compromise

The next simplest case to consider is when $D^2v(d|\boldsymbol{\lambda}')$ has the same sign for all $\big(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\big)$. This will occur for example when one of $f_2(d|\boldsymbol{\lambda}'), f_1(d|\boldsymbol{\lambda}')$ is convex and the other concave in $\big(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\big)$. In this case clearly equation(8) has no solution, two coincident solutions or two separated solutions in $\big(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\big)$. We have considered cases above when $v(d|\boldsymbol{\lambda}')$ is increasing or decreasing in d, when one or no stationary point exists in the interval of interest. Below we focus on the case when there are two different solutions.

By our differentiability conditions the two stationary points in $\big(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\big)$ a local maximum and a local minimum. Furthermore it is easy to check that in a primal conflict when $D^2v(d|\boldsymbol{\lambda}') > 0$, $d \in \big(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\big)$ and $p_1(a_1|\boldsymbol{\lambda}') = 0$ the only maxima of V are either the smaller of these two intensities or $b_1(\boldsymbol{\lambda}')$. On the other hand when $D^2v(d|\boldsymbol{\lambda}') > 0$ and $p_2(b_2|\boldsymbol{\lambda}') = 0$ the only maxima of V are either $a_2(\boldsymbol{\lambda}')$ or the larger of these two interior intensities. In these two cases we have a choice between a compromise and all out attack - in the first scenario or total focus on the campaign in the second. In the dual case we simply reverse the roles of maxima and minima in the above. Any choice between the two options largely determined by ρ. So in all these case C2 avoids some possibilities of contradiction in the commander but risks lack of contiguity.

It is often straightforward to find the solutions to (8) when the two densities $p_1(d|\boldsymbol{\lambda}'), p_2(d|\boldsymbol{\lambda}')$ have a known algebraic form. We illustrate below a boundary scenario where $v(d|\boldsymbol{\lambda}')$ satisfies the convexity conditions outlined above.

Example 2 (Zero-one utility/beta beliefs) *Consider the setting described in the example above where, for $i = 1, 2$, $P_i(d|\boldsymbol{\lambda}')$ has a beta $B(\alpha_i, \beta_i)$ density $p_i(d|\alpha_i, \beta_i)$ on the interval $d \in [0, 1] = I$ (so $a = 0$ and $b = 1$) given by*

$$\begin{aligned} p_1(d|\alpha_1, \beta_1) &= \frac{\Gamma(\alpha_1 + \beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)} d^{\beta_1 - 1}(1 - d)^{\alpha_1 - 1} \\ p_2(d|\alpha_2, \beta_2) &= \frac{\Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_2)\Gamma(\beta_2)} d^{\alpha_2 - 1}(1 - d)^{\beta_2 - 1} \end{aligned}$$

The function $V(d|\boldsymbol{\lambda})$ is then differentiable in d for $d \in (0, 1)$ so by equation(8) the commander's decision will be 1) $d = 0$ - to keep intensity to the minimum and so minimally compromise the campaign 2) $d = 1$ - to engage with full intensity in order to attain the mission with highest probability or 3) to choose a compromise decision d which satisfies

$$\tau(d|\alpha, \beta) = \alpha \log d + \beta \log(1 - d) = \rho' \tag{10}$$

where $\alpha = \alpha_2 - \beta_1$, $\beta = \beta_2 - \alpha_1$ and

$$\rho' = \rho + \rho_3(\boldsymbol{\lambda}')$$

where

$$\rho_3(\boldsymbol{\lambda}') = -\log \frac{\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2)\Gamma(\beta_2)}{\Gamma(\alpha_2 + \beta_2)\Gamma(\alpha_1)\Gamma(\beta_1)} \tag{11}$$

Note in particular that in the two types of symmetric scenarios when $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ or when $\alpha_1 = \beta_2$ and $\beta_1 = \alpha_2$ the term $\rho_3(\boldsymbol{\lambda}) = 0$ so that the parameter ρ' is exactly the daring ρ. Equation(10) implies that there are either $0, 1$ or 2 interior critical points and 0 or 1 local maximum which is a potential compromise solution as well as the two extreme intensities. We consider 4 cases in turn:

$\alpha > 0, \beta < 0$ *In this case $\tau(d|\alpha, \beta)$ is strictly increasing on $(0, 1)$ corresponds to a maximum of V. This compromise option is always better than fully committing to the mission or campaign objectives at the exclusion of the other.*

$\alpha < 0, \beta > 0$ *In this case $\tau(d|\alpha, \beta)$ is strictly decreasing on $(0, 1)$ corresponds to a minimum of V. In this situation the rational commander will choose either $d = 1$ - pure combat or $d = 0$ - pure circumspection. The actual choice with depend on the value of ρ' - the larger ρ' the more inclined the commander is to choose combat.*

$\alpha > 0, \beta > 0$ *This occurs when, for example, the maximum negative effect on the campaign of a chosen level of intensity is approached much more quickly than the effect of intensity on the success of the mission. Here it can be seen that $\tau(d|\alpha,\beta)$ has two values in $(0,1)$: the smaller a maximum and the larger a minimum of V. With large negative values of ρ' the rational commander chooses a low but non- zero value of intensity obtaining almost optimal results associated with campaign objectives but allowing small chances of the mission success which is more uncertain. As ρ' increases, for example because the mission objectives are given a higher priority then this intensity smoothly increases. However at some point before the intensity maximizing τ is reached the commander switches from the partial compromise to pure combat.*

$\alpha < 0, \beta < 0$ *This happens when for example, the maximum negative effect on the campaign of a chosen level of intensity is approached much more slowly than the effect of intensity on the success of the mission. Here again $\tau(d|\alpha,\beta)$ has two values in $(0,1)$: but this time the smaller is a minimum and the larger a maximum of V. With large negative values of ρ' the rational commander chooses pure circumspection but as ρ' increases a point where the Bayes decision suddenly switches to a moderately high intensity, this intensity smoothly increasing to pure combat as $\rho' \to \infty$.*

All scenarios where $v(d|\boldsymbol{\lambda}')$ is either strictly convex or concave exhibit an analogous geometry to the one discussed above: only the exact algebraic form of the equations governing the stationary point change. Although surprisingly common in simple examples, this convexity condition is not a generic one. It cannot model all scenarios adequately and competing decisions can often develop in subtler ways. In these cases it is necessary to use somewhat more sophisticated mathematics to understand and classify the ensuing phenomena.

3.4 Conflict and differential conditions

For the purposes of this section we make the qualitative assumption that for all values of $\lambda' \in \Lambda'$ $p_1(.|\boldsymbol{\lambda}')$ and $p_2(.|\boldsymbol{\lambda}')$ are both *unimodal* with its unique mission point mode denoted by $\xi_1(\boldsymbol{\lambda}')$ and its unique campaign point mode $\xi_2(\boldsymbol{\lambda}')$. Further assume that $p_1(.|\boldsymbol{\lambda}')$ and $p_2(.|\boldsymbol{\lambda}')$ - are continuously differentiable on the open interval $\left(a(\boldsymbol{\lambda}'), b(\boldsymbol{\lambda}')\right)$. It will then follow that

$$\begin{aligned} Df_1(d|\boldsymbol{\lambda}') &> 0 \textit{ when } a(\boldsymbol{\lambda}') \leq d < \xi_1(\boldsymbol{\lambda}') \\ Df_1(d|\boldsymbol{\lambda}') &< 0 \text{ when } \xi_1(\boldsymbol{\lambda}') < d \leq b(\boldsymbol{\lambda}') \\ Df_2(d|\boldsymbol{\lambda}') &> 0 \text{ when } a(\boldsymbol{\lambda}') \leq d < \xi_2(\boldsymbol{\lambda}') \\ Df_2(d|\boldsymbol{\lambda}') &< 0 \text{ when } \xi_2(\boldsymbol{\lambda}') < d \leq b(\lambda) \end{aligned} \tag{12}$$

We have seen in the discussion of equation(9) that when the mission point is smaller than the campaign point in a primal scenario the Bayes decisions of all rational commanders are compromises and this decision is a continuous function of the hyperparameters and this is the only scenario which is not bifurcated, We now study the complementary situation Thus suppose for a $\lambda' \in \Lambda'$ the mode $\xi_2(\boldsymbol{\lambda}') < \xi_1(\boldsymbol{\lambda}')$: i.e. the mission point is larger than the campaign point. Then when $d \in (a(\boldsymbol{\lambda}), b(\boldsymbol{\lambda})) \cap (\xi_2(\boldsymbol{\lambda}'), \xi_1(\boldsymbol{\lambda}'))$

$$Dv(d|\boldsymbol{\lambda}) = Df_2(d|\boldsymbol{\lambda}') - Df_1(d|\boldsymbol{\lambda}') < 0 \tag{13}$$

this being true independently of the value of ρ. The stationary points $d_0(\boldsymbol{\lambda})$ of V satisfy (8) so define a value ρ_0 such that

$$\rho_0(d_0(\boldsymbol{\lambda}), \boldsymbol{\lambda}') = v(d_0(\boldsymbol{\lambda})|\boldsymbol{\lambda}') = f_2(d_0(\boldsymbol{\lambda})|\boldsymbol{\lambda}') - f_1(d_0(\boldsymbol{\lambda})|\boldsymbol{\lambda}') \tag{14}$$

This implies that any choice of ρ_0 making $d_0(\rho_0, \boldsymbol{\lambda}')$ a stationary point, makes $d_0(\boldsymbol{\lambda})$ a local minimum of $V(d|(\rho_0, \boldsymbol{\lambda}'))$ and furthermore this is unique. It follows by (**??**) that in a primal scenario $V(d_0|\boldsymbol{\lambda})$ must have one local maximum $\varsigma_2(\boldsymbol{\lambda}) < \xi_2(\boldsymbol{\lambda}')$, and a local maximum $\varsigma_1(\boldsymbol{\lambda}) > \xi_1(\boldsymbol{\lambda}')$. The scenario is therefore bifurcated and will present possible problems for C2.

Since $Dv(d|\boldsymbol{\lambda}) < 0$ for *any* value of ρ for any $d \in (a(\boldsymbol{\lambda}), b(\boldsymbol{\lambda})) \cap \xi_2(\boldsymbol{\lambda}'), \xi_1(\boldsymbol{\lambda}')$ then in particular no Bayes decision can lie in this interval a phenomenon described by [17] as *inaccessibility.* In particular fixing $\boldsymbol{\lambda}'$ and running ρ from $-\infty$ to ∞. From the monotonicity property $d_0^*(\rho)$ is discontinuous in ρ at some value $\rho^*(\boldsymbol{\lambda}') : \rho_1(\boldsymbol{\lambda}') < \rho^*(\boldsymbol{\lambda}') < \rho_2(\boldsymbol{\lambda}')$. The set of optimal decisions thus bifurcates into two disjoint sets: either lying in the interval $(a(\boldsymbol{\lambda}'),\varsigma_2(\boldsymbol{\lambda}')]$ and be of "low intensity" more consistent with campaign objectives or be in $[\varsigma_1(\boldsymbol{\lambda}'),b(\boldsymbol{\lambda}'))$ and be of "high intensity" and be more consistent with mission objectives.

Thus when $\xi_2(\boldsymbol{\lambda}') < \xi_1(\boldsymbol{\lambda}')$ C2 cannot avoid a potential lack of contiguity, even in primal scenarios .Furthermore the smaller the campaign point $\xi_2(\boldsymbol{\lambda}')$ relative to the mission point $\xi_1(\boldsymbol{\lambda}')$ the larger the inaccessibility regions will tend to be and so the worse the potential lack of contiguity. So the relative position of the mission and campaign points has a critical role in the geometrical description of the resolvability of conflict for the rational commander.

4 Links to catastrophes

4.1 Catastrophes and rational choice

The bifurcation phenomenon we have described in this paper is actually a more general example of some well studied singularities, especially the cusp (and dual cusp) catastrophe, that are classified for in infinitely differentiable functions see e.g. [12], [17], [6]. Thus, for the purposes of this section assume now within the interval $d \in (a(\boldsymbol{\lambda}), b(\boldsymbol{\lambda}))$ that $V(d|\boldsymbol{\lambda})$ is infinitely differentiable in d and consider the points $(d^0, \boldsymbol{\lambda}_0) \in I \times \Lambda$ of $(d, \boldsymbol{\lambda})$ which are stationary points in this interval: i.e. that satisfy (8). On this manifold the points for which the next two derivatives of this function are zero: i.e. the parameter values $\lambda' \in \Lambda'$ of the two densities and a stationary value of d

$$Df_1(d^0|\boldsymbol{\lambda}_0') = Df_2(d^0|\boldsymbol{\lambda}_0') \tag{15}$$

are called *fold points*. If in addition we have that at that stationary value

$$D^2 f_1(d^0|\boldsymbol{\lambda}_0') = D^2 f_2(d^c|\boldsymbol{\lambda}_0') \tag{16}$$

also holds then such a $\lambda \in \Lambda$ is called a *cusp point*. These points are of special interest, because near such values $\lambda_0 \in \Lambda$ the geometry of $V(d|\boldsymbol{\lambda})$ changes. In the zero - one example these points will be largely determined by the *actual* situation faced by the commander.

An important theorem called the Classification Theorem demonstrates that for most functions V and dimensions of the non-local and scale parameters in Λ is less than 7 the way this geometry changes can be classified into a small number of shapes called catastrophes [17] each linked to the geometry of a low order polynomial. In our case the cusp points and fold points are especially illuminating because we will see below that, in many scenarios the commander's expected utility will exhibit a geometry associated with one of two of these catastrophes the cusp catastrophe in the case of primal scenarios and the dual cusp catastrophe in the dual scenario.

Suppose that Λ can be projected down on to a two dimensional subspace $C \subseteq R^2, C \subseteq \Lambda$. called the *control space*. Suppose this contains a single cusp. The cusp is a continuous curve in C with a single point $c(\boldsymbol{\lambda}_0')$ called the *cusp point* where the curve is not differentiable and turns back on itself to form a curly v shape. Points on this continuous line are called *fold points.* Their coordinates can be obtained by solving the first two equations above in $\boldsymbol{\lambda}$ and then projecting these on to C.

It is convenient to parametrize the space C using coordinates (n, s) which are oriented around this cusp. The *splitting factor* s takes a value 0 at the cusp point along the (local) line of symmetry of the cusp orientated so that positive values lie within the v. We will see below that typically in this application, in symmetric scenarios the splitting factor is increasing of the distance $\xi_2(\boldsymbol{\lambda}') - \xi_1(\boldsymbol{\lambda}')$ between the campaign and mission points of the commander's expected utility. This is however not a function of ρ and so in particular not a function of the utility weights. In this sense it is somewhat a feature of the scenario faced by the commander rather than the commander herself and so in particular a more robust feature for C2 to estimate. The *normal factor* n also takes a value 0 at the cusp point and is orthogonal to s. In our examples it is always a function of the parameter ρ as well as other features which might make the problem non-symmetric and can in principle take any value depending on the commander's criterion weights.

It has now been shown that under a variety of regularity conditions, discrete mixtures of two unimodal distribution typically exhibit at most on cusp point see e.g. [14],[15]. When $V(d|\boldsymbol{\lambda})$ exhibits a single cusp point its geometry is simple to define. For values of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $(n(\boldsymbol{\lambda}),s(\boldsymbol{\lambda}))$ lies outside the v of the cusp. there is exactly one stationary point d^* of $V(d|\boldsymbol{\lambda})$ where d is in the open interval (a, b), under the assumptions above d^* must be a local (and therefore global) maximum of $V(d|\boldsymbol{\lambda})$, and so the commander's best rational choice. In this scenario, because $d^* \in (a, b)$ this course of action can be labelled as a compromise between the two objectives. The extent to which the compromise will favour one of the two objectives will depend of the commander's current values of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ which in turn depend on his prioritization and beliefs. In this region $d^*(\boldsymbol{\lambda})$ will be continuous in $\boldsymbol{\lambda}$ and so evolve continuously as the commander's circumstances evolve.

On the other hand, for values of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that $(n(\boldsymbol{\lambda}),s(\boldsymbol{\lambda}))$ lies within the v of the cusp, under the assumptions above there will (exceptionally) be two turning points and a maximum, or (typically) two maxima, $d^*(1)$ and $d^*(2)$ and a minimum. In the latter usual scenario the commander's optimal choice will depend on the relative height of these local maxima. If the maximum $d^*(1)$ closer to a is such that $V(d^*(1)|\boldsymbol{\lambda}) > V(d^*(2)|\boldsymbol{\lambda})$ where $d^*(2)$ is the maximum closer to b then the rational commander chooses a low intensity option. If $V(d^*(1)|\boldsymbol{\lambda}) > V(d^*(2)|\boldsymbol{\lambda})$ then the rational choice is the higher intensity option. Note that this is analogous to the circumstances we have described above. In this case C2 can experience lack of contiguity and regret at least for central values of ρ.

The dual scenario - less favourable to C2 - has an identical geometry but with maxima and minima permuted. Since rational behaviour is governed by maxima, the behavioral consequences on the commander of the geometry are quite different. Outside the v of the cusp, optimal decisions are thrown on to the boundary and the scenario becomes pseudo unresolvable. On the other hand parameters inside the v of the cusp allow there to be an interior maximum of the expected utility as well as the two extreme options. Usually as we move further into the v of the cusp the relative efficacy of the interior decision improves relative to the extremes until the Bayes decision becomes a compromise decision.

Rather than dwell on these generalizations we now move on to demonstrate the geometries explicitly for some well known families of distribution.

4.2 Some illustrative examples

Example 3 (Zero -one beta catastrophe) *From the catastrophe point of view this is particularly simple. The fold points are obtained as solutions of* $D\tau = 0$ *which lie in the interior* $(0,1)$ *of the space of possible Bayes decisions. The solution in terms of* $d^* = \alpha(\alpha+\beta)^{-1}$ *lies in* $(0,1)$ *if and only if* α *and* β *are of the same sign: the last two of the four special cases we analyzed. Explicitly they are given by* $\alpha\beta > 0$ *and*

$$(\rho', \alpha, \beta) = (\rho^{f\prime}(\alpha, \beta), \alpha, \beta)$$

where

$$\rho^f(\alpha, \beta) = \alpha \log \alpha + \beta \log \beta - (\alpha+\beta)\log(\alpha+\beta)$$

It is easy to check there are no cusp points satisfying the above. Here the control space can be expressed in one dimension and this one dimensional space summarizes the geometry of the their commander's utility function, as described earlier. Once C2 identifies whether the scenario is primal or dual and whether $\alpha\beta < 0$ *or* $\alpha\beta > 0$ *the value of* $\rho^f(\alpha, \beta)$ *and whether or not the value of* $\rho' < \rho^f(\alpha, \beta)$*,* *if $\rho^f(\alpha, \beta)$ exists explains the range of possibilities. In this sense the existence and position of fold points is intrinsic to understanding the geometry. Finally note that this geometry is qualitatively stable in the sense that other utilities satisfying the same strict convexity/concavity condition illustrated in this example can never exhibit cusps and will exhibit exactly analogous geometry of projection of its singularities but be governed by different equations on different hyperparameters.*

Because this is a boundary scenario the above example is not general enough to capture all important geometries that C2 may encounter. Typically these cases include cusps. Consider the following example.

Example 4 (gamma distributions) *Suppose the distributions P_1 and P_2 are (translated) gamma distributions having log densities on $(a = 0, b = 2\bar{b})$ given by*

$$\begin{aligned} f_1(d) &= c_1 + \beta_1 m_1 \log(2\bar{b} - d) - \beta_1(2\bar{b} - d),\ d \leq 2\bar{b} \\ f_2(d) &= c_2 + \beta_2 m_2 \log d - \beta_2 d,\ d \geq 0 \end{aligned}$$

where $c_i = \alpha_i \log \beta_i - \log \Gamma(\alpha_i)$, $m_i = \beta_i^{-1}(\alpha_i - 1)$ and $\alpha_i, \beta_i > 1$ so that each density has its mode strictly within the interior of its support. The equation(3) of the stationary points of the commander's expected utility is then

$$\beta_2 m_2 \log d - \beta_2 d - \beta_1 m_1 \log(2\bar{b} - d) + \beta_1(2\bar{b} - d) = (\beta_1 + \beta_2)\rho'$$

where $\rho' = (\beta_1 + \beta_2)^{-1}(\rho + c_1 - c_2)$ Letting $\beta = \beta_1(\beta_1 + \beta_2)^{-1}$, $\delta = d - \bar{b}$ this simplifies to

$$(1 - \beta)m_2 \log(\bar{b} + \delta) - \beta m_1 \log(\bar{b} - \delta) - \delta = \rho' + (1 - 2\beta)\bar{b}$$

The modes of the two densities on δ are given by the mission point $\xi_1 = \bar{b} - m_1$ and campaign point $\xi_2 = m_2 - \bar{b}$. By differentiating with respect to δ substituting and reorganizing it follows that the fold points for δ such that $-\bar{b} < \delta < \bar{b}$ must satisfy the quadratic equation

$$\delta^2 + \left[(1 - 2\beta)\bar{b} - (\beta\xi_1 + (1 - \beta)\xi_2)\right]\delta + \bar{b}((1 - \beta)\xi_2 - \beta\xi_1) = 0$$

This scenario can therefore be identified with the canonical cusp catastrophe [17] whose fold points are also given by a quadratic. In particular its cusp points satisfy

$$\delta = \left[(1 - 2\beta)\bar{b} - (\beta\xi_1 + (1 - \beta)\xi_2)\right]$$

The fold points exist when

$$\left[(1 - 2\beta)\bar{b} - (\beta\xi_1 + (1 - \beta)\xi_2)\right]^2 \geq 4\bar{b}((1 - \beta)\xi_2 - \beta\xi_1)$$

Note that when $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta'$ so that $\beta = 1/2$ and $\xi_1 = -\xi_2$ this simplifies to there being fold points only when $\xi_2 \leq \xi_1$ and a cusp point at

$(\delta, \xi_1, \xi_2) = (0,0,0)$. *This is consistent with the results concerning inaccessibility discussed after (12) and the two competing decisions get further apart as* ξ_1 *and* $\overline{b}$ *increase since the fold points are given by* $\delta = \pm\sqrt{\overline{b}\xi_1}$ *with inaccessible decisions between these two values.*

Example 5 (dual gamma) *In the dual scenario to the one described above the cusp point defines the emergence of a compromise solution whilst pure circumspection* $a = 0$ *and pure combat* $b = 2\overline{b}$ *are always competing local maxima of the expected utility. However as the modes* ξ_1 *of* Q_2 *and* ξ_2 *of* Q_1 *become increasingly separated the compromise region grows and becomes the Bayes decision of most commanders.*

Although being able to identify this phenomenon with a canonical cusp/dual cusp catastrophe as above is unusual, for many pairs of candidate distribution the most complicated singularity we encounter is usually a cusp catastrophe. Thus consider the following example.

Example 6 (Weibull distributions) *Let* X *have an exponential distribution with distribution function* $1 - \exp -1/2x$ *and suppose that the distribution functions* P_1 *and* P_2 *are the distributions of* $X_1 = 2\left(\sigma^{-1}\left\{b - X\right\}\right)^c$ *and* $X_2 = 2\left(\sigma^{-1}\left\{X + a\right\}\right)^c$ *so that for* $d \in (a(\boldsymbol{\lambda}), b(\boldsymbol{\lambda}))$, $a(\boldsymbol{\lambda}) < b(\boldsymbol{\lambda})$ *the respective densities on this interval are given by*

$$\begin{aligned} p_1(d) &= e^r (b-d)^{c-1} \exp\left\{-\frac{1}{2}\left\{\sigma^{-1}(b-d)\right\}^c\right\} \\ p_2(d) &= e^r (d+a)^{c-1} \exp\left\{-\frac{1}{2}\left\{\sigma^{-1}(d+a)\right\}^c\right\} \end{aligned}$$

Here $\sigma > 0$ *and for simplicity we will assume* $0 < c \leq 2$. *Note that when* $c > 1$, *the densities are unimodal with mission point* $\xi_1(\boldsymbol{\lambda}) = b - \sigma\left\{2(1-c^{-1})\right\}^{1/c}$ *and campaign point* $\xi_2(\boldsymbol{\lambda}) = a + \sigma\left\{2(1-c^{-1})\right\}^{1/c}$ *and stationary points satisfy*

$$f_2(\delta) - f_1(\delta) = (c-1)\left[\log\left(\delta + \overline{b}\right) - \log\left(\overline{b} - \delta\right)\right] - \frac{1}{2}\sigma^{-c}\left[\left(\delta + \overline{b}\right)^c - \left(\overline{b} - \delta\right)^c\right] = \rho \tag{17}$$

where $\overline{b} = 1/2(b-a)$ *and* $\delta = d - 1/2(a+b)$ *- so that* $-\overline{b} \leq \delta \leq \overline{b}$. *Differentiating and rearranging this expression when* $c \neq 1$ *decisions on the fold points must also satisfy*

$$\psi(\delta^2, \overline{b}) \triangleq \left(\overline{b}^2 - \delta^2\right)\phi(\delta^2, \overline{b}) = g \tag{18}$$

where $g \triangleq 2(1-c^{-1})\sigma^c$ *and*

$$\phi(\delta^2, \overline{b}) = \frac{1}{2}\overline{b}^{-1}\left\{\left(\overline{b} + \delta\right)^{c-1} + \left(\overline{b} - \delta\right)^{c-1}\right\} > 0$$

Note that when $1 < c < 2$, $\phi(\delta^2, \overline{b})$ is strictly decreasing in δ^2. The cusp points also need to satisfy

$$\begin{aligned} D\psi(\delta^2, \overline{b}) &= 0 \Leftrightarrow \\ (c-1)\left(\overline{b}^2 - \delta^2\right)\left\{\left(\overline{b}+\delta\right)^{c-2} - \left(\overline{b}-\delta\right)^{c-2}\right\} &= 2\delta\left\{\left(\overline{b}+\delta\right)^{c-1} + \left(\overline{b}-\delta\right)^{c-1}\right\} \end{aligned}$$

Note in particular that for each value of $\overline{b}$ there is always a cusp point at $(\delta, \rho, g) = (0, 0, \overline{b}^c)$ and the splitting factor of such a cusp is $\overline{b}^c - g$:: largest when the difference between the campaign point and mission point is large and when the uncertainty σ is small. When $0 < c < 1$, $g < 0$ but $\psi(\delta^2, \overline{b}) > 0$ so no fold points exist. As ρ increases the best course of action jumps when $\rho = 0$ from pure circumspection a to the value b of pure combat. When $c = 1$ the stationary points are given by those value, unique functions of the parameters satisfying $\delta_c = d_c - 1/2(a+b) = \rho/\sigma$ and this again is always a minimum except when $\rho = 0$ when all intensities in $[a, b]$ are equally good. Finally when $1 < c \leq 2$ because ψ is decreasing in δ^2 and $g > 0$, there is a single pair of stationary points $(-\delta^*, \delta^*)$ - coinciding when $\delta^* = 0$ - lying on fold points if and only if

$$\psi(0) = \overline{b}^c = (1/2(b-a))^c \geq \left(2(1 - c^{-1})\sigma^c\right) = g$$

It can easily be checked that for a given $\overline{b}$ there is a single cusp point at $(\delta, \rho, g) = (0, 0, \overline{b}^c)$. In the special case when $c = 2$, the fold points are given by

$$\delta^2 = \overline{b}^2 - \sigma^2$$

There are therefore no fold points if $\overline{b}^2 = 1/4(b-a)^2 < \sigma^2$ whilst if $1/4(b-a)^2 \geq \sigma^2$ the fold points are given by

$$d = 1/2(a+b) \pm \sqrt{1/4(b-a)^2 - \sigma^2}$$

Differentiating and solving gives that the cusp point satisfies

$$1/4(b-a)^2 = \sigma^2, d = 1/2(a+b)$$

The distance between the campaign and mission point is therefore again central here. See [[14]] for further analyses of the geometry of this special case and its generic analogues. Note that this case is used to explain and categorize the results of two battle group exercises we discuss in [3].

Like in the gamma example above the assumption of equality in the uncertainty parameter for the two distributions is not critical in the example above in the sense that the underlying geometry can still be described in terms of s continuum of cusp points and details of their exact coordinates for the case $c = 2$ can be found in [15].. It turns out the richest geometry is obtain in the equal variance case, and when the uncertainty associated with one of the objectives is much higher than the other the large uncertainty objective tends to get ignored in favour of the other and the problem tends to degenerate

We end by elaborating the first example to analyse the geometry of non-boundary scenarios of this type. We note that as we move away for the boundary, cusp catastrophes like those appearing in the last two example are exhibited in this example as well.

Example 7 (General Beta Case) *For $i = 1, 2$, let $P_i(d|\boldsymbol{\lambda}')$ be the density of $2X_i - 1 + (-1)^i c$ where X_1 has a beta $B(\alpha_i, \beta_i)$ density given in the earlier example and $|c| \leq 1$. Then $I(\boldsymbol{\lambda}') = [|c| - 1, 1 - |c|]$ and the scenario is primal when $c > 0$, dual when $c < 0$ whilst when $c = 0$ we have a linear transformation of the boundary case of the last example. Writing $\gamma_i = \alpha_i - 1, \epsilon_i = \beta_i - 1$, $i = 1, 2$. The equation (8) becomes*

$$\gamma_2 \log(1 + d + c) + \epsilon_2 \log(1 - d - c) - \gamma_1 \log(1 + d - c) - \epsilon_1 \log(1 - d + c) = \rho''$$

where

$$\rho'' = \rho + \sum_{i=1,2} \log \Gamma(\alpha_i) + \log \Gamma(\beta_i) - \log \Gamma(\alpha_i + \beta_i) + (\gamma_2 + \epsilon_2 - \gamma_1 - \epsilon_1) \log 2$$

Differentiating and reorganizing we find that in the fold points in $I(\boldsymbol{\lambda}')$ must satisfy the cubic

$$\sum_{j=0}^{3} c_j d^j = 0$$

where

$$\begin{aligned} c_0 &= (1 - c^2)[(\epsilon_1 + \gamma_2)(1 - c) - (\gamma_1 + \epsilon_2)(1 + c)] \\ c_1 &= (\epsilon_1 - \gamma_2)(1 - c)^2 - (\gamma_1 - \epsilon_2)(1 + c)^2 \\ c_2 &= -(\epsilon_1 + \gamma_2)(1 + c) + (\gamma_1 + \epsilon_2)(1 - c) \\ c_3 &= \gamma_2 + \epsilon_2 - \gamma_1 - \epsilon_1 \end{aligned}$$

This situation is therefore slightly more complicated than the boundary on we discussed earlier, because there is the possibility that two *local and potentially competing maxima appear in the interior of $I(\boldsymbol{\lambda}')$. However when a commander is comparably certain of the effect of chosen intensity on mission and campaign objectives then $\gamma_1 + \epsilon_1 = \gamma_2 + \epsilon_2$ the fold point becomes quadratic and we recover the geometry of the single canonical cusp/dual cusp catastrophe. After a little algebra the cusp points related to the modes through the equation.*

$$\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1} = c^2$$

When $c = 0$ - our earlier case - this equation degenerates into requiring $P_1 = P_2$ - but otherwise such cusp points exist and are feasible whenever $\xi_2 > \xi_1$. This demonstrates how our original example can be generalized straightforwardly away form convexity to a situation where compromise appears as an expression of the cusp catastrophe.

5 Discussion

An agile C2 can draw several conclusions from this analysis about how to organize, train and communicate intent and freedoms for decision-making to commanders: a number of these conclusions already being accepted as good practice. Here we will assume that commanders face a scenario where both P_1 and P_2 are twice differentiable and unimodal.

1. Whenever appropriate and possible mission statements and campaign objectives should be stated in such a way that they are resolvable so that well-trained rational commanders can safely compromise.

2. When a scenario cannot be presented as resolvable then, if planning to devolve decision making C2 should aim to present a commander with a pseudo resolvable scenario. The first of two conditions required for this is that the scenario is primal. This means that the commander can perfectly address the campaign objectives whilst still having some possibility of completing the mission to some degree of success and there is a level of intensity ideal for attaining mission objective which also can be expected not to totally jeopardize the campaign. It will often be possible to make a scenario primal simply by the way the two objectives are communicated to the commander. The second requirement is to control the mission and campaign points modes so that the intensity with the best incremental improvement on mission success occurs at a value ensuring maximal campaign integrity and the best incremental improvement on campaign success occurs at a value ensuring maximal mission. A rational commander will then choose to compromise between the two objectives. The actual compromise point will depend on each commander's individual training and emotional history but the careful matching of contiguous commanders should ensure coherence.

3. When neither of the two scenarios described above are achievable then in most cases, provided the mission point is lower than the campaign point, the devolved commander can still be expected to compromise and not to be faced with contradiction. In this case C2 arrangements must be prepared to expect lower levels of contiguity but coherence can still be managed by carefully considering the commanders' capacity to deal with stresses. In particular to encourage compromise C2 should try to ensure that mission statements allow for there to be an option which scores at least half as well as the best option for mission and at least half as well for campaign objectives. Note that if it is made clear that partially achieved success in the two objectives is more highly rated then the likelihood of compromise is increased.

4. Problems of lack of contiguity and contradiction can be expected to occur if the mission point is much higher than the campaign point. If C2 still plans to devolve in these cases then they must endeavor to keep the distance

between the mission and campaign points as small as possible since this will limit the extent of the discontiguity and contradiction (see the analysis of the last section).

5. The most undesirable scenarios are those that are unresolvable or pseudo unresolvable. In these cases the focus falls on ρ and therefore, unless the intensity associated with pure combat is close to that for pur circumspection, the training, deployment and personality of individual commanders will become criticl. The C2 arrangements are then most stable if a top-down style is adopted.

All these points rely on the assumption of commander rationality. In [1], [3] we detail results from two experiments studying how experienced personnel respond to conflicting scenarios. The first was a mission where there was high risk of casualties. The second was a potential threat to a civilian convoy where the commander had to balance the efficacy of defence from attack and a negotiated passage. Participants were then encouraged to document their decision processes. The commanders often reasoned differently but interestingly all choose courses of action consistent with the rationality described above. Perhaps one of the most interesting findings was that confidence in succeeding in the objectives - mainly reflected in the choice of ρ - had a big influence on course of action selection. Conclusions from these experiments, aided by the implementation of the ideas above have informed procurement of command information systems [13]. Of course in real time a commander can only evaluate a few possible courses of action [11], ,[9], [10] but we argue in [3] that this does not invalidate the approach above, it just approximates it. So both from the theoretical and practical perspective this rational model - where C2 assumes its commanders choose what is rationally consistent with their individual nature, experience and competencies is a good starting point for C2 arrangements and their organization for training and selection.

References

[1] Dodd, L. Moffat, J Smith, J.Q. and Mathieson. G.(2003) "From simple prescriptive to complex descriptive models: an example from a recent command decision experiment" *Proceedings of the 8th International Command and Control Research and Technology Symposium* June Washington

[2] Dodd, L.(1997), "*Command decision studies for future conflict"* DERA Unpublished Report.

[3] Dodd, L. Moffat, J. and Smith, J.Q.(2006) "*Discontinuities in decision-making when objective conflict: a milatary command decsision case study"* J.Oper. Res. Soc.,57, p.643 - 654

[4] von Winterfeldt, D. and Edwards, W. (1986) "*Decision Analysis and behavioral research models"*, Cambridge Univ Press

[5] French, S. and Rios Insua, D.(2000) "*Statistical Decision Theory*" Arnold

[6] Harrison, P.J. and Smith, J.Q. (1980) "Discontinuity, decision and conflict"(with discussion) *Bayesian Statistics 1* Eds Bernado et al p 99- 142.

[7] Keeney, R.L. and Raiffa, H. (1976) "*Decisions with Multiple Objectives*" Wiley New York

[8] Janis, J.L. and Mann, L.(1977) *"Decision Making: A Psychological Analysis of Conflict, Choice and Commitment "*Free Press. N.Y.

[9] Moffat, J. (2002) *"Command and Control in the Information Age"* The Stationary Office, London

[10] Moffat,J. and Witty, S. (2002) "Bayesian Decision Making and military control" *J. of Operations Research Society , 53, 709 - 718*

[11] Perry, W. and Moffat, J, (2004) "Information sharing among military headquarters: the effects on decision making" MG-226-UK, The RAND Corperation Santa Monica, CA,USAMeasuring the effects of knowledge in military campaigns" *J. of Operations Research Society , 48, 10 , 965-72*

[12] Poston, T. and Stewart, I. "*Catastrophe Theory and its applications"* Pitman

[13] Saunders, M.J. and Miles, J. (2004) "How can network enabled capability contribute to better command and control" Proceedings of the 9th ICCRTS Copenhagen

[14] Smith, J.Q.(1979) "Mixture Catastrophes and Bayes decision theory" *Math Proc.Camb.Phil.Soc.*86, 91-101

[15] Smith, J.Q. (1983) "Catastrophe Theory: A way of seeing" *University College London Research Report*

[16] Smith, J.Q., Harrison, P.J., and Zeeman, E.C.(1981) "The analysis of some discontinuos decision Processes" *European Journnal of Operations Research* vol 7 30-43.

[17] Zeeman E.C.(1977) *"Catastrophe Theory: Selected Papers"* Addison Wesley

6 Appendix

Writing $u[0, \boldsymbol{\lambda}] = \inf_{d \in D}\{\overline{U}(\boldsymbol{d}|\boldsymbol{\lambda})\}$ and $u[1, \boldsymbol{\lambda}] = \sup_{d \in D}\{\overline{U}_i(\boldsymbol{d}|\boldsymbol{\lambda})\}$, to obtain (3) note that $\overline{U}(\boldsymbol{d}|\boldsymbol{\lambda})$ is an increasing linear transformation of $\alpha_1^0(\boldsymbol{\lambda})U_1^0(\boldsymbol{d}|\boldsymbol{\lambda}) + \alpha_2^0(\boldsymbol{\lambda})U_2^0(\boldsymbol{d}|\boldsymbol{\lambda})$ where, for $i = 1, 2$, $U^0(\boldsymbol{d}|\boldsymbol{\lambda}) = \left\{U_i^0(d, \boldsymbol{d}_i^*(d)|\boldsymbol{\lambda}) - u[0, \boldsymbol{\lambda}]\right\}\{u[1, \boldsymbol{\lambda}] - u[0, \boldsymbol{\lambda}]\}^{-1}$, $U_i^0(\boldsymbol{d}|\boldsymbol{\lambda}) = \left\{\overline{U}_i(\boldsymbol{d}|\boldsymbol{\lambda}) - u[0, \boldsymbol{\lambda}]\right\}\{u_i[1, \boldsymbol{\lambda}] - u[0, \boldsymbol{\lambda}]\}^{-1}$and $\alpha_i^0(\boldsymbol{\lambda}) = \alpha_i(\boldsymbol{\lambda})u_i[1, \boldsymbol{\lambda}]u[1, \boldsymbol{\lambda}]^{-1}$.Note that these renormalizations simply ensure that $\alpha_1^0(\boldsymbol{\lambda}) + \alpha_2^0(\boldsymbol{\lambda}) = 1$, $\sup U^0(\boldsymbol{d}|\boldsymbol{\lambda}) =$

$\sup U_1^0(\boldsymbol{d}|\boldsymbol{\lambda}) = \sup U_2^0(\boldsymbol{d}|\boldsymbol{\lambda}) = 1$ and $\inf U^0(\boldsymbol{d}|\boldsymbol{\lambda}) = \inf U_1^0(\boldsymbol{d}|\boldsymbol{\lambda}) = \inf U_2^0(\boldsymbol{d}|\boldsymbol{\lambda}) = 0$. For each fixed value of d a rational commander chooses the decision $\boldsymbol{d}_i^*(d)$ maximizing $U_i'(\boldsymbol{d}|\boldsymbol{\lambda})$, $i = 1, 2$ - respectively and then chooses d so as to maximize

$$\alpha_1^0(\boldsymbol{\lambda}) P_1(d|\boldsymbol{\lambda}) + \alpha_2^0(\boldsymbol{\lambda})\left(1 - P_2(d|\boldsymbol{\lambda})\right)$$

where$P_1(d|\boldsymbol{\lambda}) = U_1^0(d, \boldsymbol{d}_1^*(d)|\boldsymbol{\lambda})$ and $P_2(d|\boldsymbol{\lambda}) = 1 - U_1^0(d, \boldsymbol{d}_2^*(d)|\boldsymbol{\lambda})$. On substitution this can be seen to be maximized when $V(d|\boldsymbol{\lambda})$ of equation(3).is maximized.

Theorem 8 *If $V(d_0^*(\boldsymbol{\lambda}), \boldsymbol{\lambda})$ is continuous in d at all values $\boldsymbol{\lambda} \in \Lambda$ and $d^*(\boldsymbol{\lambda}_0)$, defined above, is unique and there exists, for a fixed value of $\boldsymbol{\lambda}_0$, an $\eta' > 0$ such that $V(d, \boldsymbol{\lambda}_0)$ is strictly increasing in d when $d^*(\boldsymbol{\lambda}_0) - \eta' < d < d^*(\boldsymbol{\lambda}_0)$ and strictly decreasing when $d^*(\boldsymbol{\lambda}_0) < d < d^*(\boldsymbol{\lambda}_0) + \eta'$, then $d^*(\boldsymbol{\lambda}_0)$ is continuous in $\boldsymbol{\lambda}$ at $\boldsymbol{\lambda}_0$.*

Proof. For $\delta > 0$, let $V^*(\boldsymbol{\lambda}_0) = \sup_{d \in D}\{V(d, \boldsymbol{\lambda}_0)\}$and $A(\boldsymbol{\lambda}, \delta(\eta')) = \{d : V(d, \boldsymbol{\lambda}) \geq V^*(\boldsymbol{\lambda}) - \delta\}$ where

$$\delta(\eta') = \max\{V(d^*(\boldsymbol{\lambda}_0), \boldsymbol{\lambda}_0) - V(d^*(\boldsymbol{\lambda}_0) - \eta', \boldsymbol{\lambda}_0), V(d^*(\boldsymbol{\lambda}_0), \boldsymbol{\lambda}_0) - V(d^*(\boldsymbol{\lambda}_0) + \eta', \boldsymbol{\lambda}_0)\}$$

Then, from the uniqueness of $d^*(\boldsymbol{\lambda}_0)$ and the monotonicity conditions above, for all $\varepsilon > 0$, there exists an $\eta'(\varepsilon) > 0$ such that $A(\boldsymbol{\lambda}_0, \delta(\eta')) \subseteq B(d^*(\boldsymbol{\lambda}_0), \varepsilon(\eta'))$ where $B(d^*(\boldsymbol{\lambda}_0), \varepsilon)$ is an open ball centred at $d^*(\boldsymbol{\lambda}_0)$ of radius ε. By the continuity of $V(d, \boldsymbol{\lambda}_0)$ at $(d^*(\boldsymbol{\lambda}_0), \boldsymbol{\lambda}_0)$, for all $\varepsilon > 0$ there exists an $\eta(\omega) > 0$ such that if $\| \lambda_0 - \lambda\|_0 < \eta$ then $|V^*(\boldsymbol{\lambda}_0) - V^*(\boldsymbol{\lambda})| < \varepsilon$. Thus

$$\begin{aligned} d^*(\boldsymbol{\lambda}) &\in A(\boldsymbol{\lambda}, \delta) \subseteq \{d : V(d, \boldsymbol{\lambda}_0) > V^*(\boldsymbol{\lambda}) - \delta - \omega\} \\ &= A(\boldsymbol{\lambda}_0, \delta + \omega) \subseteq B(d^*(\boldsymbol{\lambda}_0), 2\varepsilon) \end{aligned}$$

which implies that, for all $\varepsilon > 0$ there is an $\eta''(\varepsilon) = \min[\eta'(\varepsilon), \eta(\omega)] > 0$ such that if $\| \boldsymbol{\lambda} - \boldsymbol{\lambda}_0\|_0 < \eta$, $|d^*(\boldsymbol{\lambda}) - d^*(\boldsymbol{\lambda}_0)| < 2\varepsilon$: i.e. $d^*(\boldsymbol{\lambda})$ is continuous at $\boldsymbol{\lambda}_0$. ■