A SIMPLE PROOF OF KRAMKOV'S RESULT ON UNIFORM SUPERMARTINGALE DECOMPOSITIONS

SAUL JACKA [1]
University of Warwick

Abstract: we give a simple proof of Kramkov's uniform optional decomposition in the case where the class of density processes satisfies a suitable closure property. In this case the decomposition is previsible.

Keywords: UNIFORM SUPERMARTINGALE; UNIFORM OPTIONAL DECOMPOSITION; UNIFORM PREDICTABLE DECOMPOSITION

AMS subject classification: 60G15

§1 Introduction

In [4], Kramkov showed that for a suitable class of probability measures, $\mathcal{P}$, on a filtered measure space $(\Omega, \mathcal{F}, \mathcal{F}_t; t \geq 0)$, if S is a supermartingale under all $\mathbb{Q} \in \mathcal{P}$, then there is a uniform optional decomposition of S into the difference between a $\mathcal{P}$-uniform local martingale and an increasing optional process. In this note we give (in Theorem 2.2) a simple proof of this result in the case where the martingale logarithms of the density processes of the p.m.s in $\mathcal{P}$ (taken with respect to a suitable reference p.m.) are closed under scalar multiplication (and hence continuous).

The applications in [4] refer to the financial set-up, where $\mathcal{P}$ is the collection of Equivalent Martingale Measures for a collection of discounted securities $\mathcal{X}$, and S is the payoff to a superhedging problem for an American option, so that

$$S_t = \operatorname*{ess\ sup}_{\mathbb{Q} \in \mathcal{P}} \operatorname*{ess\ sup}_{\text{stopping times } \tau \geq t} \mathbb{E}[X_\tau | \mathcal{F}_t],$$

where X is the claims process for the option.

Other examples are a multi-period coherent risk-measure where the risk measure ρ_t is given by

$$\rho_t(X) = \operatorname*{ess\ sup}_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}[X | \mathcal{F}_t]$$

(see [4]) and the Girsanov approach to a control set-up, where S is given by the same formula, but $\mathcal{P}$ corresponds to a collection of costless controls on X (see, for example, [1]).

§2 Uniform supermartingale decomposition

We assume that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, satisfying the usual conditions, and a collection, $\mathcal{P}$, of probability measures on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \sim \mathbb{P}$, for all $\mathbb{Q} \in \mathcal{P}$.

We note that, since $\mathbb{Q} \sim \mathbb{P}$, $\Lambda^{\mathbb{Q}}_t \stackrel{def}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ is a positive $\mathbb{P}$-martingale, with $\Lambda^{\mathbb{Q}}_0 = 1$.

Lemma 2.1. We may write $\Lambda^{\mathbb{Q}}_t = \mathcal{E}(\lambda^{\mathbb{Q}})_t$, where $\mathcal{E}$ is the Doleans-Dade exponential and $\lambda^{\mathbb{Q}}_t = \int_0^t \frac{d\Lambda^{\mathbb{Q}}{}_s}{\Lambda^{\mathbb{Q}}_{s-}}$, so that $\lambda^{\mathbb{Q}}$ is a $\mathbb{P}$-local martingale with jumps strictly bounded below by -1.

[1] I am most grateful to an anonymous referee for pointing out an error in an earlier draft of this paper and for very helpful comments on the presentation

Proof From Theorem II.8.3 of [4], if neither $\Lambda^{\mathbb{Q}}_t$ nor $\Lambda^{\mathbb{Q}}_{t-}$ vanishes then $\lambda^{\mathbb{Q}}$ exists. The fact that $\Lambda^{\mathbb{Q}}$ does not vanish follows from the stronger statement that, since $\mathbb{Q} \sim \mathbb{P}$, $\Lambda^{\mathbb{Q}}$ is strictly positive. This also implies, once we have established its existence, the condition on the jumps of $\lambda^{\mathbb{Q}}$.

The fact that $\Lambda^{\mathbb{Q}}_{t-}$ does not vanish follows via the following argument. First note that $\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t} = \big(\Lambda^{\mathbb{Q}}_t\big)^{-1}$, so $\big(\Lambda^{\mathbb{Q}}_t\big)^{-1}$ is a $\mathbb{Q}$-martingale. Now $\mathcal{M}^{loc}$, the class of local martingales, is equal to H^{loc}_1, the localisation of $H_1 = \{\text{martingales } M : \mathbb{E}[\sup_{0\le t<\infty}|M_t|] < \infty\}$ [see [2]). So, it follows that there is a localising sequence (T_n) of stopping times increasing ($\mathbb{Q}$ and hence) $\mathbb{P}$-a.s.to ∞ such that $\mathbb{E}\big[\sup_{t\le T_n}\big(\Lambda^{\mathbb{Q}}_t\big)^{-1}\big] < \infty$. It follows from the integrability that $\sup_{t\le T_n}\big(\Lambda^{\mathbb{Q}}_t\big)^{-1}$ is $\mathbb{P}$-a.s. finite and hence $\inf_{t\le T_n}\Lambda^{\mathbb{Q}}_t$ is $\mathbb{P}$-a.s. strictly positive. Thus $\Lambda^{\mathbb{Q}}_{t-}$ is $\mathbb{P}$-almost surely positive on the stochastic interval $[[0,T_n]]$. Now letting $n\uparrow\infty$ we see that the second requirement is satisfied. □

We denote by $\mathcal{L}$ the collection $\{\lambda^{\mathbb{Q}};\ \mathbb{Q}\in\mathcal{P}\}$ and by $\mathcal{L}^{loc}$ the usual localisation of $\mathcal{L}$.

Theorem 2.2 *Suppose that*

i) $\mathbb{P}\in\mathcal{P}$; *and*

ii) $\mathcal{L}^{loc}$ *is closed under scalar multiplication;*

then any $\mathcal{P}$*-uniform non-negative supermartingale,* S*, possesses a class-uniform Doob-Meyer predictable decomposition, i.e. we may write* S *uniquely as*

$$S = M - A,$$

where M *is a* $\mathcal{P}$*-uniform local martingale and* A *is a locally integrable predictable increasing process with* $A_0 = 0$*.*

Remark: Notice that condition (ii) implies that every element of $\mathcal{L}^{loc}$ is continuous. This follows since: any element of $\mathcal{L}^{loc}$ has jumps bounded below by -1; then if $\delta\lambda\in\mathcal{L}^{loc}$ for all $\delta\in\mathbb{R}$, by taking appropriately large positive and negative values of δ, we see that the jumps of λ must be of size zero.

Proof of Theorem 2.2: take $\mathbb{Q}\in\mathcal{P}$, with $\Lambda^{\mathbb{Q}} = \mathcal{E}(\lambda^{\mathbb{Q}})$. Now S is a non-negative $\mathbb{Q}$-supermartingale iff $S\Lambda^{\mathbb{Q}}$ is a non-negative $\mathbb{P}$-supermartingale so, taking the Doob-Meyer decomposition of S with respect to $\mathbb{P}$: $S = M - A$, we must have that

$$\begin{aligned} S\Lambda^{\mathbb{Q}} &= S_0 + \int S_{t-}d\Lambda^{\mathbb{Q}}_t + \int \Lambda^{\mathbb{Q}}_t dS_t + < S, \Lambda^{\mathbb{Q}} > \\ &= S_0 + \int S_{t-}d\Lambda^{\mathbb{Q}}_t + \int \Lambda^{\mathbb{Q}}_t dM_t + \int \Lambda^{\mathbb{Q}}_t (d < \lambda^{\mathbb{Q}}, M >_t - dA_t) \end{aligned} \tag{2.2}$$

is a $\mathbb{P}$-supermartingale. Now since the first two terms in the last line of (2.2) are local martingales, whilst the last is a predictable process of integrable variation on compacts, it follows that the last term must be decreasing.

Now we claim that we must then have

$$< \lambda^{\mathbb{Q}}, M >^{+} << A, \text{ with } \frac{d < \lambda^{\mathbb{Q}}, M >^{+}}{dA} \le 1, \tag{2.2}$$

where $< \lambda^{\mathbb{Q}}, M >^{+}$ and $< \lambda^{\mathbb{Q}}, M >^{-}$ are, respectively, the increasing processes corresponding to the positive and negative components in the Hahn decomposition of the signed measure induced by $< \lambda^{\mathbb{Q}}, M >$.

This follows from the more general statement: if μ, m^+ and m^- are three σ-finite measures on a measurable space $(\Omega, \mathcal{F})$ and

(i) m^+ and m^- are mutually singular

and

(ii) $\nu \stackrel{def}{=} \mu - m^+ + m^-$ is also a measure, then

$$m^+ << \mu, \text{ with } \frac{dm^+}{d\mu} \leq 1.$$

To see this, take $A \in \mathcal{F}$ with $m^+(A) > 0$ then $m^-(A) = 0$ and $\nu(A) = \mu(A) - m^+(A) \geq 0$ so $\mu(A) > 0$ and $m^+(A) \leq \mu(A)$.

Now $\mathcal{L}^{loc}$ is closed under scalar multiplication so that, localising if necessary, we may assume that $\delta\lambda \in \mathcal{L}$ and so, defining $\mathbb{Q}^\delta$ by $\Lambda^{\mathbb{Q}^\delta} \stackrel{def}{=} \mathcal{E}(\delta\lambda^{\mathbb{Q}})$, we see that (2.2) holds with $\lambda^{\mathbb{Q}}$ replaced by $\delta\lambda^{\mathbb{Q}}$ for any $\delta \in \mathbb{R}$. Letting $\delta \to \infty$ we see that $\frac{d<\lambda^{\mathbb{Q}},M>^+}{dA} = 0$, whilst letting $\delta \to -\infty$ we see that $\frac{d<\lambda^{\mathbb{Q}},M>^-}{dA} = 0$. It follows immediately that

$$< \lambda^{\mathbb{Q}}, M > \equiv 0$$

To complete the proof we need simply observe that

$$M\Lambda^{\mathbb{Q}} = M_0 + \int M_{t-} d\Lambda^{\mathbb{Q}}_t + \int \Lambda^{\mathbb{Q}}_t dM_t + \int \Lambda^{\mathbb{Q}}_t d < M, \lambda^{\mathbb{Q}} >_t,$$

and hence M is a $\mathbb{Q}$-local martingale and since $\mathbb{Q}$ is arbitrary, the result follows □

Remark: We note that if $\mathcal{P}$ consists of the EMMs for a vector-valued martingale M and the underlying filtration supports only continuous martingales (for example if it is the filtration of a multi-dimensional Wiener process), then the conditions of Theorem 2.2 are satisfied. This follows since, under these conditions, if λ is a $\mathbb{P}$-local martingale then $\lambda \in \mathcal{L}^{loc} \Leftrightarrow < \lambda, M >= 0$, and the same then holds for any multiple of λ.

REFERENCES

[1] Benes, V: Existence of Optimal Stochastic Control Laws (1971), *SIAM J. of Control and Optim.* **9**, 446-472.

[2] Jacod, J: Calcul Stochastique et Problèmes de Martingales, LNM 714, Springer, Berlin (1979) .

[3] Jacod, J and Shiryaev, A: Limit Theorems for Stochastic Processes *2nd Edition*, Springer, Berlin, (2003).

[4] Kramkov, D: Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets (1996), *Prob. Th & Rel. Fields* **105**, 459-479.

[5] Reidel, F: Dynamic Coherent Risk Measures (2004), *Stoch. Proc. and Appl.* **112**, 185-200.

DEPARTMENT OF STATISTICS
UNIVERSITY OF WARWICK
COVENTRY
CV4 7AL
UK
s.d.jacka@warwick.ac.uk