Canonical correlations for dependent gamma processes

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Abstract: The present paper provides a characterization of exchangeable pairs of random measures (μ_1, μ_2) whose identical margins are fixed to coincide with the distribution of a gamma completely random measure, and whose dependence structure is given in terms of canonical correlations. It is first shown that canonical correlation sequences for the finite-dimensional distributions of (μ_1, μ_2) are moments of means of a Dirichlet process having random base measure. A few related illustrations are provided, with some of them being of interest for applications to Bayesian statistics. Necessary and sufficient conditions are further given for canonically correlated gamma completely random measures to have independent joint increments. Finally, time-homogeneous Feller processes with gamma reversible measure and canonical autocorrelations are characterized as subordinated Dawson–Watanabe diffusions with independent homogeneous immigration. It is thus shown that such Dawson–Watanabe diffusions subordinated by pure drift are the only processes in this class whose time-finite-dimensional distributions have, jointly, independent increments.

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1. Introduction

The definition of probability distributions on $\mathbb{R}^d$, with fixed margins, has a long history. An approach that stands out for its elegance, and the wealth of mathematical results it yields, is due to H.O. Lancaster (see [18]) who used

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orthogonal functions on the marginal distributions. This can be summarized in a formulation that is strictly related to the purpose of the present paper as follows. Suppose π_1 and π_2 are two probability measures on $\mathbb{R}^d$ and let $\{Q_{1,\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}_+^d\}$ and $\{Q_{2,\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}_+^d\}_{n=1}^{\infty}$ be collections of multivariate functions that form complete orthonormal sets on π_1 and π_2, respectively. Moreover, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be random vectors defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and taking values in $\mathbb{R}^d$ with probability distribution π_1 and π_2, respectively. If for every $\boldsymbol{m}$ and $\boldsymbol{n}$ in $\mathbb{Z}_+^d$ one has

$$\mathbb{E}\left[Q_{1,\boldsymbol{n}}(\boldsymbol{X})Q_{2,\boldsymbol{m}}(\boldsymbol{Y})\right] = \rho_{\boldsymbol{n}}\, \delta_{\boldsymbol{n},\boldsymbol{m}}, \tag{1}$$

with $\delta_{\boldsymbol{n},\boldsymbol{m}} = 1$ if $\boldsymbol{n} = \boldsymbol{m}$ and 0 otherwise, then X and Y are said to be canonically correlated, and $\{\rho_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}_+^d\}$ is called the *canonical correlation sequence* of $(\boldsymbol{X}, \boldsymbol{Y})$. For $d = 1$ a complete characterization of the set of all possible canonical correlation sequences for bivariate distributions with gamma marginals is given in [12]. In this case the Laguerre polynomials define a family of orthogonal functions.

In the present paper we aim at characterizing canonical correlations for pairs of random measures (hence, of infinite-dimensional random elements), with identical marginal distributions given by the law of a gamma completely random measure. To this end, we first need to introduce some notation and point out a few definitions. Let $\mathbb{X}$ be a complete and separable metric space and $\mathscr{X}$ the Borel σ-algebra on $\mathbb{X}$. Set $M_{\mathbb{X}}$ as the space of boundedly finite measures on $(\mathbb{X}, \mathscr{X})$ equipped with a Borel σ–algebra $\mathscr{M}_{\mathbb{X}}$ (see [4] for details). A *completely random measure* (CRM) $\widetilde{\mu}$ is a measurable mapping from $(\Omega, \mathscr{F}, \mathbb{P})$ into $(M_{\mathbb{X}}, \mathscr{M}_{\mathbb{X}})$ such that, for any two disjoint sets A and B, the random variables $\widetilde{\mu}(A)$ and $\widetilde{\mu}(B)$ are independent. In particular, $\widetilde{\mu}$ is a *gamma* CRM with parameter measure cP_0 if

$$\phi_{(c,P_0)}(f) := \mathbb{E}\left[\mathrm{e}^{-\widetilde{\mu}(f)}\right] = \exp\left\{-c\int_{\mathbb{X}} \log(1 + f(x))\, P_0(\mathrm{d}x)\right\}$$

for any measurable function $f : \mathbb{X} \to \mathbb{R}$ such that $\int \log(1 + |f|)\, \mathrm{d}P_0 < \infty$, where $c > 0$ and P_0 is a probability measure on $\mathbb{X}$. Henceforth we will use the notation Γ_{cP_0} when referring to it. Furthermore, P_0 is assumed non–atomic.

In section 4 we will define a pair of random measures $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ to be in canonical correlation if all its finite-dimensional distributions have canonical correlations. The distributional properties of any such pair, when each $\widetilde{\mu}_i$ is a gamma CRM with identical parameter, will be investigated by resorting to the Laplace functional transform

$$\phi(f_1, f_2) = \mathbb{E}\left[\mathrm{e}^{-\widetilde{\mu}_1(f_1) - \widetilde{\mu}_2(f_2)}\right] \tag{2}$$

for any pair of measurable functions $f_i : \mathbb{X} \to \mathbb{R}$ such that $\int \log(1 + |f_i|)\, \mathrm{d}P_0 < \infty$, for $i = 1, 2$. It will be shown that a vector $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ of gamma CRMs has canonical correlations if and only if its joint Laplace functional has the following representation:

$$\frac{\phi(f_1, f_2)}{\phi_{(c,P_0)}(f_1)\, \phi_{(c,P_0)}(f_2)} = \mathbb{E}\left[\exp\left\{\int_{\mathbb{X}} \frac{f_1 f_2}{(1 + f_1)(1 + f_2)}\, \mathrm{d}K\right\}\right] \tag{3}$$

for some random measure K dominated by a CRM $\widetilde{\mu}$ which is an independent copy of $\widetilde{\mu}_i$ $(i = 1, 2)$. We will also find out a surprising fact: the canonical correlations associated to the finite–dimensional distributions of $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ can be expressed as mixed moments of linear functionals of Dirichlet processes. This connects our work to various, and seemingly unrelated, areas of research where means of Dirichlet processes play an important role. See [20] far a detailed account. Using orthogonal functions expansions, along with the interpretation as moments of Dirichlet random means, simplifies the analytic treatment of canonically correlated gamma measures, and lead to easy algorithms for sampling from their joint distribution. We will describe such algorithms and show that they can all be thought of as generalizations of a neat Gibbs sampling scheme proposed in [6] for a class of one-dimensional gamma canonical correlations. This analysis also allows us to identify the form of canonical correlations that yield a vector $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ being still CRM, i.e. such that for any pair of disjoint sets A and B in $\mathscr{X}$ the random vectors $(\widetilde{\mu}_1(A), \widetilde{\mu}_2(A))$ and $(\widetilde{\mu}_1(B), \widetilde{\mu}_2(B))$ are independent.

Our investigation is, finally, extended to encompass a characterization of canonical (auto)correlations for continuous time, time-homogeneous and reversible measure-valued Markov processes with gamma stationary distribution. We will use, once again, the connection with Dirichlet random means to prove that all the Markov processes in this class can be derived via a Bochner-type subordination of a well-known measure-valued branching diffusion process with immigration (its generator is recalled in Section 5, (19)), to which we will refer as the Γ-Dawson-Watanabe process. We also prove that the Γ-Dawson-Watanabe process is the only instance in this class with the property that all its time-bidimensional distributions are, jointly, completely random measures. We also use our algorithms and a result of [11] on Γ-Dawson-Watanabe process, to derive a simple expansion for the transition function of general stationary gamma processes with canonical autocorrelations.

1.1. Motivations

One of the reasons that motivated our interest into the subject stems from a rich body of literature in Bayesian statistical inference aiming at describing and investigating dependent random probability measures. These better fit situations where the data are subject to forms of dependence more general than exchangeability. Indeed, in many applications involving, e.g., nonparametric regression, meta-analysis, two-sample problems, and so on, the usual exchangeable assumption appears inadequate whereas it is more natural to consider the observations as partially exchangeable. To be more specific, let $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ be two sequences of $\mathbb{X}$–valued random elements defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Set $P_{\mathbb{X}}$ as the space of probability measures on $(\mathbb{X}, \mathscr{X})$ endowed with the topology of weak convergence. A typical situation occurring in many statistical applications is compatible with the following characterization of the finite–dimensional distributions of the processes $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$

$$\mathbb{P}\left[\boldsymbol{X}^{(n_1)} \in \times_{i=1}^{n_1} A_i,\ \boldsymbol{Y}^{(n_2)} \in \times_{i=1}^{n_2} B_i \mid (\widetilde{p}_1, \widetilde{p}_2)\right] = \prod_{i=1}^{n_1} \widetilde{p}_1(A_i) \prod_{j=1}^{n_2} \widetilde{p}_2(B_j).$$

where $\boldsymbol{X}^{(n_1)} = (X_1, \ldots, X_{n_1})$, $\boldsymbol{Y}^{(n_2)} = (Y_1, \ldots, Y_{n_2})$, $\{A_i\}_{i=1}^{n_1}$ and $\{B_j\}_{j=1}^{n_2}$ are collections of sets in $\mathscr{X}$. The vector $(\widetilde{p}_1, \widetilde{p}_2)$ takes values in $P_{\mathbb{X}}^2$ and its distribution is denoted as Q. In the last decade there has been a wealth of proposals of Q in this setting and most of them rely on variations of the so–called stick–breaking representation of the marginals $\widetilde{p}_1$ and $\widetilde{p}_2$. Nice recent surveys on the subject are provided in [7] and in [24].

Here we rely on an alternative approach that makes use of CRMs, $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$. Hence, if each $\widetilde{p}_i$ is some function of $\widetilde{\mu}_i$, for $i = 1, 2$, the dependence between $\widetilde{p}_1$ and $\widetilde{p}_2$ is determined by the dependence between $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$. Possible options for defining dependent completely random measure have already been investigated in [14, 9, 19, 21]. The approach we are undertaking, based on canonical correlations, has various merits: (i) it captures various forms of dependence between $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ and, hence, $\widetilde{p}_1$ and $\widetilde{p}_2$; (ii) it leads to the determination of analytic results in closed form and algorithms for simulation, that are of interest for the evaluation of various quantities of statistical interest; (iii) it allows to deduce a number of distributional properties beyond the second order moments; (iv) investigating time-dependent canonical correlations for measure-valued Markov processes can be helpful e.g. in Bayesian nonparametric regression, where the time-parameter plays the role of a covariate.

1.2. Outline of the paper

The outline of the paper is as follows. In Section 2 we recall come background on canonical correlations for random variables and specialize the discussion to the gamma case which involves the use of Laguerre polynomials. In Section 3 we move on to considering canonically correlated vectors with independent gamma random components and provide a result that paves the way to the extension to CRMs we are aiming at. An algorithm for simulating such vectors is also described and it can be seen that it is easily implementable. In Section 4 we define canonically correlated CRMs and provide a characterization in terms of their joint Laplace functional and in terms of moments of Dirichlet random means. Section 5 discusses several notable examples corresponding to different specifications of the canonical correlations, all driven by a measure K being a "doubly-stochastic"variant of the so-called extended gamma process (see e.g. [8]). Finally, Section 6 characterizes time–homogenous and reversible Markov processes with values in $M_{\mathbb{X}}$ with canonical autocorrelations.
The proofs of all the original results are deferred to the Appendices, unless otherwise stated.

2. Canonical correlations for gamma random variables

Canonical correlations were originally defined via orthogonal polynomials, and firstly applied in Statistics in [18]. Let π be a probability measure on $\mathbb{R}^d$ and $\{Q_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}_+^d\}$ a system of orthonormal polynomials with weight measure π indexed by their degree $\boldsymbol{n} \in \mathbb{Z}_+$. Hence $\int Q_{\boldsymbol{n}} Q_{\boldsymbol{m}} \,\mathrm{d}\pi = \delta_{\boldsymbol{n},\boldsymbol{m}}$. It can be easily shown that any two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, sharing the same marginal distribution π, are canonically correlated as in (1), with canonical correlation sequence $\{\rho_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}_+^d\}$, if and only if

$$\mathbb{E}\left[Q_{\boldsymbol{n}}(\boldsymbol{X}) \mid \boldsymbol{Y} = \boldsymbol{y}\right] = \rho_{\boldsymbol{n}}\, Q_{\boldsymbol{n}}(\boldsymbol{y}) \quad \boldsymbol{n} \in \mathbb{Z}_+^d, \tag{4}$$

for every $\boldsymbol{y}$ in the support of π. In this conditions, since any function f such that $\int_{\mathbb{R}^d} f^2 \,\mathrm{d}\pi < \infty$ has a series representation:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{Z}_+^d}^{\infty} \widehat{f}(\boldsymbol{n}) Q_{\boldsymbol{n}}(\boldsymbol{x})$$

where $\widehat{f}(\boldsymbol{n}) := \mathbb{E}\left[f(\boldsymbol{X}) Q_{\boldsymbol{n}}(\boldsymbol{X})\right]$, for any $\boldsymbol{n} \in \mathbb{Z}_+^d$, then one has

$$\mathbb{E}\left[f(\boldsymbol{Y}) \mid \boldsymbol{X} = \boldsymbol{x}\right] = \sum_{\boldsymbol{n} \in \mathbb{Z}_+^d} \widehat{f}(\boldsymbol{n}) \rho_{\boldsymbol{n}} Q_{\boldsymbol{n}}(\boldsymbol{x}).$$

Note that $\rho_{\boldsymbol{0}}$ must be equal to 1 in order, for the conditional expectation operator, to map constant functions to constant functions. The canonical correlation coefficients ρ thus contain all the information about the dependence between $\boldsymbol{X}$ and $\boldsymbol{Y}$. At the extremes, *"$\rho_{\boldsymbol{n}}$ for every $\boldsymbol{n} \in \mathbb{Z}_+^d \setminus \{\boldsymbol{0}\}$"* implies independence and *"$\rho_{\boldsymbol{n}} = 1$ for every $\boldsymbol{n} \in \mathbb{Z}_+^d$"* corresponds to perfect dependence. Expansions in L^2 can be easily determined for the joint and conditional Laplace transforms as soon as an appropriate generating function for the orthogonal polynomials is available, as in the Gamma case.

We now focus on the case where $d = 1$ and π is a gamma distribution, namely

$$\pi(\mathrm{d}x) = \Gamma_{\alpha,\beta}(\mathrm{d}x) = \frac{1}{\beta^\alpha \Gamma(\alpha)}\, x^{\alpha-1}\, \mathrm{e}^{-\frac{x}{\beta}}\, \mathrm{d}x\, \mathbb{1}_{(0,\infty)}(x)$$

where $\alpha > 0$ and $\beta > 0$. It is worth recalling a few well–known facts about gamma random variables in canonical correlation, mostly owed to several works of Eagleson and Griffiths (see for example [10, 12, 13]).

The Laplace transform of a $\Gamma_{\alpha,\beta}$ distribution is denoted as $\phi_{\alpha,\beta}(t) = (1 + \beta t)^{-\alpha}$. With no loss of generality in the sequel we maintain the assumpion $\beta = 1$, unless otherwise stated.

A collection of orthogonal polynomials with respect to $\Gamma_{\alpha,1}$ is represented by the Laguerre polynomials

$$L_{n,\alpha}(x) = \frac{(\alpha)_n}{n!}\, {}_1F_1(-n; \alpha; x)$$

for $\alpha > 0$, where ${}_1F_1$ is the confluent hypergeometric function and $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ is the n–th ascending factorial of α. It can be easily checked that

$$\int_{\mathbb{R}^+} L_{n,\alpha}(x) L_{m,\alpha}(x)\, \Gamma_{\alpha,1}(\mathrm{d}x) = \frac{(\alpha)_n}{n!}\, \delta_{n,m}.$$

Hence, a collection of orthonormal polynomials $\{L^*_{n,\alpha} : n \geq 1\}$ can be defined by setting $L^*_{n,\alpha} = (n!/(\alpha_n))^{1/2}\, L_{n,\alpha}$. Finally, Laguerre polynomials with leading coefficient equal to unity will be henceforth denoted as $\widetilde{L}_{n,\alpha}$, where $\widetilde{L}_{n,\alpha} = n!(-1)^n\, L_{n,\alpha}$. From this definition one finds out that

$$c_{n,\alpha} := \mathbb{E}\left[\widetilde{L}_{n,\alpha}(X)^2\right] = n!\,(\alpha)_n. \tag{5}$$

Under this scaling, closed form expressions can be determined for the generating functions

$$G_\alpha(r,x) := \sum_{n=1}^{\infty} \widetilde{L}_{n,\alpha}(x) \frac{r^n}{n!} = (1+r)^{-\alpha} e^{\frac{xr}{1+r}} \quad |r| < 1 \tag{6}$$

and the Laplace transform

$$\psi_{n,\alpha}(t) := \int_0^\infty e^{-tx} \widetilde{L}_{n,\alpha}(x) \Gamma_{\alpha,1}(\mathrm{d}x) = (\alpha)_n \left(\frac{-t}{t+1}\right)^n (1+t)^{-\alpha} \tag{7}$$

See [10]. An important result we refer to was proved in [12] and is as follows.

Theorem 1 (Griffiths [12])**.** *A sequence $(\rho_n)_{n\geq 0}$ is a sequence of canonical correlation coefficients of the law of a pair (X,Y) with identical marginals $\Gamma_{\alpha,1}$ if and only if $\rho_n = \mathbb{E}[Z^n]$, for a random variable Z whose distribution has support* $[0,1]$.

Theorem 1 implies that the set of all gamma canonical correlation sequences is convex and its extreme points are in the set $\{(z^n)_{n\geq 1} : z \in [0,1]\}$. For any $z \in [0,1]$, set $\mathbb{E}_z$ as the expected value computed with respect to the probability distribution of (X,Y) associated to the "extreme" canonical correlation sequence $(z^n)_{n\geq 1}$. Then

$$\begin{aligned} \phi_z(s,t) = \mathbb{E}_z\left[\mathrm{e}^{-sX-tY}\right] &= (1+s)^{-\alpha}(1+t)^{-\alpha} \sum_{n\geq 0} \frac{(\alpha)_n}{n!}\, z^n\, \theta^n \\ &= \phi_{\alpha,1}(s)\, \phi_{\alpha,1}(t)\, \phi_{\alpha,z}(-\theta) \end{aligned} \tag{8}$$

where $\theta = st(1+s)^{-1}(1+t)^{-1}$. The corresponding density is well-known (see e.g. [1], formula (6.2.25), p. 288) to be

$$p_z(\mathrm{d}x,\mathrm{d}y) = \Gamma_{\alpha,1}(\mathrm{d}x)\Gamma_{\alpha,1}(\mathrm{d}y) \frac{e^{-\frac{(x+y)z}{1-z}}}{(xyz)^{\frac{\alpha-1}{2}}} I_{\alpha-1}\left(\frac{2(xyz)^{\frac{1}{2}}}{1-z}\right), \tag{9}$$

where I_c is the modified Bessel function of order c, $c > -1$. One can use (7) and (6) to deduce a simple form for the conditional Laplace transform of Y given X in the extreme family, namely

$$\begin{aligned}\phi_z(s|x) &:= \mathbb{E}_z\left[\mathrm{e}^{-sY} \mid X = x\right] = (1+s)^{-\alpha} G_\alpha\left(\frac{-sz}{1+s}, x\right) \\ &= \phi_{\alpha,1}(s(1-z))\,\mathrm{e}^{-x\frac{sz}{1+s(1-z)}}. \end{aligned} \tag{10}$$

Thus, for general gamma canonical correlation sequences $\rho = (\rho_n)_{n\geq 0}$, by Theorem 1 the corresponding joint Laplace transforms is a mixture of extreme Laplace transforms $\phi_\rho(s,t) = \mathbb{E}\left[\phi_Z(s,t)\right]$, where Z is the random variable defining ρ via $\rho_n = \mathbb{E}\left[Z^n\right]$.

A simple algorithm is available to simulate from the joint distribution of a pair (X, Y) of $\Gamma_{\alpha,1}$ random variables with canonical correlation in the extreme family (i.e. of the form $\rho_n = z^n$, $n \in \mathbb{Z}_+$, $z \in (0,1)$). After re-scaling, the algorithm reduces to a version of a Gibbs-sampling scheme (the so-called Poisson/Gamma θ-chain) proposed by Diaconis, Khare and Saloff-Coste ([6], Proposition 4.6) to build one-step Markov kernels with $\Gamma_{\alpha,\beta}$ stationary measure and orthogonal polynomial eigenfunctions.

Algorithm A.1

(0) Initialize by choosing $\alpha > 0$ and $b \in [0, \infty)$
(i) Generate a sample $X = x$ from the $\Gamma_{\alpha,1}$ measure
(ii) Generate $N = n$ from a Poisson distribution with mean bx
(iii) Generate Y from $\Gamma_{\alpha+n,(1+b)^{-1}}$

We thus have:

Lemma 1. *The pair (X, Y) generated by Algorithm A.1 is exchangeable, with identical $\Gamma_{\alpha,1}$ marginal distributions and canonical correlation coefficients of the form $\rho_n = z^n$, $n \in \mathbb{Z}_+$ where $z = b/(1+b)$. Conversely, every bivariate gamma pair with canonical correlations of the form $\rho_n = z^n$, $n \in \mathbb{Z}_+, z \in (0,1)$, can be generated via Algorithm A.1 with $b = z/(1-z)$.*

If we denote $U = bX$ and $V = bY$, where (X, Y) is the pair generated by Algorithm A.1, then (U, V) is a bivariate pair with identical $\Gamma_{\alpha,b}$ marginal laws. Furthermore, the distribution of V conditional on $U = u$ coincides with the one-step transition distribution of Diaconis *et al.*'s θ-chain. See [6], formula (4.7), and compare it to (31) in Appendix A. Thus one can think of canonical correlations in the extreme family as the one-step correlations obtained by re-scaling the values of the θ-chain by a factor of $(1-z)/z$.

In the light of Theorem 1, algorithm A.1 can be extended in a way that it incorporates a randomization of b: the resulting variable, denoted as B, has some probability distribution P^* on $\mathbb{R}_+$ and is independent of X and Y. One, then, runs algorithm A.1 conditional on $B = b$ and the output will still be a set of realizations of a canonically correlated vector (X, Y) with $\Gamma_{\alpha,1}$ marginals. This is the key idea at the basis of Algorithm A.2 proposed in the next section to generate general canonically correlated gamma vectors in d dimensions.

3. Canonical correlations and algorithms for gamma product measures

In the present Section we derive a characterization of canonical correlations for pairs of random vectors with independent gamma coordinates, extending Griffiths' result recalled in Theorem 1. It is a preliminary step for stating the main result that will be discussed in the next Section. We will, then, focus on d–dimensional vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ having the same marginal distribution, $\pi_1 = \pi_2 = \Gamma_{\boldsymbol{\alpha},1} := \times_{i=1}^{d} \Gamma_{\alpha_i,1}$. Hence, canonical correlations are determined by (1) with

$$Q_{1,\boldsymbol{n}}(\boldsymbol{x}) = Q_{2,\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^{d} \frac{\widetilde{L}_{n_i,\alpha_i}(x_i)}{\sqrt{c_{n_i,\alpha_i}}}$$

where $\widetilde{L}_{n_i,\alpha_i}$ stands for the Laguerre polynomial of degree n_i with leading coefficient of unity and constant of orthogonality c_{n_i,α_i} as described in Section 2, and we shall henceforth use the short notation $\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) = \prod_{i=1}^{d} \widetilde{L}_{n_i,\alpha_i}(x_i)$ for any $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d$. The leading coefficient is equal to one and $c_{\boldsymbol{n},\boldsymbol{\alpha}} := \mathbb{E}[\widetilde{L}^2_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{X})] = \prod_{i=1}^{d} n_i!\,(\alpha_i)_{n_i}$. A key for generating canonically correlated gamma random vectors is provided by the following algorithm.

Algorithm A.2

- `(0)` Initialize by setting $\boldsymbol{\alpha} \in \mathbb{R}_+^d$ and a probability distribution P^* on $\mathbb{R}_+^d$
- `(i)` Generate a sample $\boldsymbol{B} = \boldsymbol{b}$ from P^*
- `(ii)` Generate d parallel runs of Algorithm A.1`(i)-(iii)` whereby, for every $j = 1, \ldots, d$, the j-th run is initialized by (α_j, b_j).
- `(iii)` Return X_j, Y_j for any $j = 1, \ldots, d$.

According to the next statement, the above algorithm does indeed characterize canonically correlated gamma random vectors and further provides a description of the canonical correlations.

Theorem 2. *Let $(\boldsymbol{X}, \boldsymbol{Y})$ be a pair of random vectors in $\mathbb{R}_+^d$ with identical marginal distribution $\Gamma_{\boldsymbol{\alpha},1}$, with $\boldsymbol{\alpha} \in \mathbb{R}_+^d$. Then $(\boldsymbol{X}, \boldsymbol{Y})$ has canonical correlations if and only if it can be generated via* Algorithm A.2 *for some probability measure P^* on $\mathbb{R}_+^d$. The canonical correlation coefficients are of the form*

$$\rho_{\boldsymbol{n}} = \mathbb{E}\left[\prod_{i=1}^{d} Z_i^{n_i}\right] \tag{11}$$

where $Z_i = B_i/(1 + B_i)$, $i = 1, \ldots, d$, and the vector $\boldsymbol{B} = (B_1, \ldots, B_d)$ has distribution P^*.

In Algorithm A.2, given $\boldsymbol{X} = \boldsymbol{x}$ and $\boldsymbol{B} = \boldsymbol{b}$, the random variables $Y_1, \ldots, Y_d$ are independent and the distribution of Y_j is a mixture of gamma distributions.

Consider the particular case whereby, in Algorithm A.2, P^* is degenerate on $\mathbb{R}_+$ so that with probability 1 one has $\boldsymbol{B} = (b, \dots, b) \in \mathbb{R}_+^d$. Since d independent Poisson random variables, conditioned to their total sum, form a multinomial vector, then an algorithm for simulating realizations of $\boldsymbol{Y}$ conditional on $\boldsymbol{X} = \boldsymbol{x}$ and $\boldsymbol{B} = (b, \dots, b)$ is

Algorithm A.3.

- (0) Initialize by fixing $b \in \mathbb{R}_+, \boldsymbol{x} \in \mathbb{R}_+^d,\ \boldsymbol{\alpha} \in \mathbb{R}_+^d$.
- (i) Sample $N = n$ from a Poisson distribution with parameter $b\,|\boldsymbol{x}|$.
- (ii) Sample $N_1 = n_1, \dots, N_d = n_d$ from a multinomial distribution with parameter vector $(n;\, x_1/|\boldsymbol{x}|, \dots, x_d/|\boldsymbol{x}|)$
- (iii) Sample $\boldsymbol{Y} = \boldsymbol{y}$ from $\Gamma_{\boldsymbol{\alpha}+\boldsymbol{n},\,(1+b)^{-1}} = \times_{i=1}^d \Gamma_{\alpha_i+n_i,(1+b)^{-1}}$

This class has an important infinite-dimensional extension, which has to do with the transition density expansion of the Γ-Dawson-Watanabe processes mentioned in the Introduction. It will be studied later in Sections 5.1 and 7.3.

4. Dependent gamma CRMs

Let us now move on to characterizing dependent vectors of gamma CRMs that are in canonical correlation. Since we are now dealing with infinite–dimensional objects, we need to make precise what we mean by a vector of CRMs in canonical correlation. To simplify notation, for any collection $\mathcal{A} = \{A_1, \dots, A_d\}$ of disjoint measurable subsets of $\mathbb{X}$ we set $\tilde{\boldsymbol{\mu}}_{j,\mathcal{A}} := (\tilde{\mu}_j(A_1), \dots, \tilde{\mu}_j(A_d))$, for each $j = 1, 2$. Denote by $\mathscr{X}^*$ the union of all finite collections of disjoint Borel sets of X, and set $\mathbb{Z}_+^* = \bigcup_{d\geq 1} \mathbb{Z}_+^d$. Throughout this and the next Sections the symbol $\mathcal{A}$ will be used for a generic element of $\mathscr{X}^*$.

Definition 1. *A pair $(\tilde{\mu}_1, \tilde{\mu}_2)$ of random measures, with $\tilde{\mu}_1 \stackrel{d}{=} \tilde{\mu}_2$, is in canonical correlation if, for every $\mathcal{A} \in \mathscr{X}^*$ the vector $\tilde{\boldsymbol{\mu}}_{1,\mathcal{A}}$ and $\tilde{\boldsymbol{\mu}}_{2,\mathcal{A}}$ are canonically correlated. The corresponding canonical correlation sequences in (1) shall be denoted denoted as $\rho := \{\rho_{\boldsymbol{n}}(\mathcal{A}) :\ \boldsymbol{n} \in \mathbb{Z}_+^*, \mathcal{A} \in \mathscr{X}^*\}$.*

Notice that the finite-dimensional distributions for non-disjoint sets can always be expressed in terms of (polynomials of) finite-dimensional distributions on disjoint sets, therefore, by linear extension, Definition 1 fully describes the dependence between the two gamma CRMs. Furthermore, notice that Definition 1 involves any choice of disjoint Borel sets $(A_1, \dots, A_d)$ but it is sufficient to check it only for collections such that $P_0(A_i) > 0,\ i = 0, \dots, d$. Indeed, if one of the sets, A_i say, has P_0 null measure, then $\tilde{L}_{cP_0(A_i),n} = \delta_{n,0}$ and therefore

$$\rho_{n_1,\dots,n_d}(A_1, \dots, A_d) = \rho_{n_1,\dots,n_{i-1},n_{i+1},\dots,n_d}(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_d).$$

We are now in a position to state then main result of this section.

Theorem 3. *Let $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ be a vector of gamma CRMs with parameter measure cP_0. Then the following conditions are equivalent:*

(i) *$(\widetilde{\mu}_1, \widetilde{\mu}_2)$ is in canonical correlation. The corresponding sequence of canonical correlations is such that for every $d \geq 1$ and collection $\mathcal{A} = \{A_1, \ldots, A_d\}$ of pairwise disjoint sets in $\mathscr{X}$*

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \mathbb{E}\left[\prod_{i=1}^{d} Z_{A_i}^{n_i}\right] \tag{12}$$

where the Z_{A_i}'s are non–negative random variables bounded by 1. Moreover, for any disjoint pair $A, B \in \mathscr{X}$,

$$Z_{A\cup B} \overset{d}{=} \varepsilon_{A,B} Z_A + (1 - \varepsilon_{A,B}) Z_B \tag{13}$$

with $\varepsilon_{A,B} \sim$ beta$(cP_0(A), cP_0(B))$ and is independent from $\{Z_A, Z_B\}$.

(ii) *The Laplace functional of $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ is as in (3), for a random measure K on $(\mathbb{X}, \mathscr{X})$ such that if $\widetilde{\mu}$ is a gamma CRM with parameter measure cP_0, then $\mathbb{P}[K(A) \geq x] \leq \mathbb{P}[\widetilde{\mu}(A) \geq x]$ for any $A \in \mathscr{X}$ and $x \in \mathbb{X}$.*

Part (iii) of Theorem 3 is reminiscent of a similar result of Griffiths and Milne ([14], formula (12)) for a class of dependent Poisson processes with identical marginal intensity, say cP_0 with the difference that, in the Poisson case, the random measure K driving the dependence was dominated by the fixed marginal intensity measure cP_0 (see [14], Section 3), whereas in our case the dominating measure is a gamma CRM, centered on cP_0.
Property (13) in part (i) sheds more light on the nature of the canonical correlation sequences for the finite-dimensional distributions of any bivariate gamma pair. This aspect will be further investigated in the remaining part of the Section.

Corollary 1. *Let G be a measure on $(x, \mathscr{X})$ such that $G(A) \leq cP_0(A)$ for any A in $\mathscr{X}$. Set independent collections $\mathscr{B} := \{B_A : A \in \mathscr{X}\}$ and $\mathscr{V} := \{V_A : A \in \mathscr{X}\}$ of $[0,1]$–valued random variables such that the variables in $\mathscr{B}$ are mutually independent and, for every $A \in \mathscr{X}$, B_A has a beta distribution with parameters $(G(A), cP_0(A) - G(A))$(with the convention that $B_A \equiv 1$ if $G(A) = cP_0(A)$). If $Z_A \overset{d}{=} B_A V_A$ for any $A \in \mathscr{X}$, then*

$$\rho_{\boldsymbol{n}}(\mathcal{A}; cP_0) = \mathbb{E}\left[\prod_{i=1}^{d} Z_{A_i}^{n_i}\right],$$

forms the canonical correlation sequence for a pair of Γ_{cP_0} CRMs if and only if

$$\rho_{\boldsymbol{n}}(\mathcal{A}; G) = \mathbb{E}\left[\prod_{i=1}^{d} V_{A_i}^{n_i}\right]$$

forms the canonical correlation sequence of a pair of Γ_G CRMs, for any $d \geq 1$, $\mathcal{A} = \{A_1, \ldots, A_d\}$ and $\boldsymbol{n} \in \mathbb{Z}_+^d$.

Another nice and surprising by-product of Theorem 3(i), is a connection between canonical correlations in (12) and linear functionals of a Dirichlet process. Indeed, if for any $d \geq 1$ the collection $\{A_1, \ldots, A_d\}$ is a measurable partition of $A \in \mathscr{X}$, then (13) can be trivially extended to

$$Z_A \overset{d}{=} \sum_{i=1}^{d} \varepsilon_i \, Z_{A_i} \qquad \text{with} \qquad \varepsilon_i = \frac{\widetilde{\mu}(A_i)}{\widetilde{\mu}(A)} \tag{14}$$

so that the distribution of $(\varepsilon_1, \ldots, \varepsilon_d)$ is a d–variate Dirichlet with parameters $(\alpha_1, \ldots, \alpha_d)$ and $\alpha_i = cP_0(A_i)$. This leads to show the following

Theorem 4. *A collection of* $[0,1]$*–valued random variables* $\{Z_A : A \in \mathscr{X}\}$ *satisfies* (13) *if and only if*

$$Z_A \overset{d}{=} \int_0^1 x \; \widetilde{p}_{\tau_A}(\mathrm{d}x) \ \ \forall A \in \mathscr{X}, \tag{15}$$

where

(i) *for any* $A \in \mathscr{X}$, τ_A *is a random measure on* $([0,1], \mathscr{B}[0,1])$ *such that* $\tau_A[0,1] = cP_0(A)$, $\tau_\varnothing = 0$ *(a.s.) and* $\tau_{A \cup B} \overset{d}{=} \tau_A + \tau_B$ *for every* $A, B \in \mathscr{X}$ *such that* $A \cap B = \emptyset$*;*

(ii) *for any* $A, B \in \mathscr{X}$ *such that* $A \cap B = \varnothing$, $\widetilde{p}_{\tau_A}$ *and* $\widetilde{p}_{\tau_B}$ *are conditionally independent Dirichlet processes on* $([0,1], \mathscr{B}[0,1])$, *given* τ_A *and* τ_B.

The distributional representation in (15) is very useful since it provides a way of computing explicitly the canonical correlations of a vector of dependent gamma processes. On the basis of (15) and of a result in [25] one can provide a combinatorial expansion for $\rho_{\boldsymbol{n}}$. The proof of the following is a direct application of [25], Proposition 2 (with the caveat that here we are dealing with random baseline measures).

Corollary 2. *For any collection* $\mathcal{A} = \{A_1, \ldots, A_d\}$ *of pairwise disjoint sets in* $\mathscr{X}$, *the canonical correlations* $\rho = \{\rho_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}_+^d, \ d \geq 1\}$ *of a vectors* $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ *of gamma CRMs can be represented as*

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \frac{1}{\prod_{i=1}^{d} (c)_{n_i}} \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} \mathbb{E}\left[\prod_{i=1}^{d} B_{n_i,k_i}(r_{1,i}, 2! r_{2,i}, \ldots, (n_i - k_i + 1)! r_{n-k+1,i}) \right] \tag{16}$$

where $r_{j,i} = c \int_0^1 x^j \, \tau_{A_i}(\mathrm{d}x)$ *and* $B_{n,k}$ *is, for any* $k \leq n$, *the partial exponential Bell polynomial.*

5. Illustrations

It is apparent from the previous results that the dependence between $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ is determined by K and, from Theorem 4 and (14), on each $A \in \mathscr{X}$ one has, marginally, $K(A) \stackrel{d}{=} \int_{[0,1]} x \, \mathrm{d}\widetilde{\mu}_{\tau_A}(dx)$ where, conditional on a random measure τ_A as in Theorem 4(i)–(ii), $\widetilde{\mu}_{\tau_A}$ is a Γ_{τ_A} CRM on $[0,1]$.

In this section we shall consider a broad class of examples of canonically correlated gamma CRMs that are generated by a random measure K on $(\mathbb{X}, \mathscr{X})$ such that, for every bounded measurable function f, $K(f) \stackrel{d}{=} \int f\zeta \mathrm{d}\widetilde{\mu}$, where $\widetilde{\mu}$ is a Γ_G process, $\zeta : \mathbb{X}\times\Omega \to [0,1]$ is some measurable function independent of $\widetilde{\mu}$ and G is a measure on $\mathbb{X}$ such that $G(A) \le cP_0(A)$ for any A in $\mathscr{X}$. Conditional on ζ, measures K of this form are known as *weighted gamma measures* or *extended gamma measures* (see [8]) with parameters (ζ, G). The corresponding bivariate gamma CRMs have thus a joint Laplace transform of the form

$$\phi(f,g) = \phi_{cP_0}(f)\,\phi_{cP_0}(g)\,\mathbb{E}\left[\mathrm{e}^{\int \log\left(1+\zeta\,\frac{fg}{(1+f)(1+g)}\right)\mathrm{d}G}\right]. \tag{17}$$

The canonical correlation sequences are derived from an expansion of (17) with both f and g as simple functions. In particular, by Theorem (3), they are determined by joint moments of random variables of the form:

$$Z_A \stackrel{d}{=} B_A \int_A \zeta(x)\,\widetilde{p}_{G,A}(\mathrm{d}x), \qquad i = 1,\dots,d \tag{18}$$

where $\widetilde{p}_{G,A}$ is a Dirichlet process with parameter measure G restricted to set A, and B_A is a beta$(G(A), cP_0(A) - G(A))$ random variable independent of ζ and $\widetilde{p}_{G,A}$. Corollary 1 guarantees that the sequence $\{Z_A : A \in \mathscr{X}\}$ satisfies (13).

Here below we consider various examples of canonically correlated gamma CRMs corresponding to different specifications of the parameters (ζ, G) for the extended gamma random measure K.

5.1. *ζ degenerate at a deterministic constant, and Dawson-Watanabe processes*

The simplest example one might think of corresponds to assuming $K = z\widetilde{\mu}$ for some constant $z \in (0,1)$. This is the same as saying $G = cP_0$ and $\zeta = z$ on $\mathbb{X}$, a.s.–$\mathbb{P}$. It is then obvious that $Z_A = z$, a.s.–$\mathbb{P}$, for every A in $\mathscr{X}$ and $\rho_{\boldsymbol{n}}(\mathcal{A}) = z^{|\boldsymbol{n}|}$ for every collection $\mathcal{A} = \{A_1,\dots,A_d\}$ of d disjoint and measurable subsets of $\mathbb{X}$ and for any vector of non–negative integers $\boldsymbol{n} = (n_1,\dots,n_d)$ with $|\boldsymbol{n}| = \sum_{i=1}^d n_i$. For $\mathbb{X}$ locally compact, the resulting distribution of the pair $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ has the interpretation as the distribution of $(\lambda\xi_{\lambda,0}, \lambda\xi_{\lambda,t})$ for any $\lambda \in \mathbb{R}_+$, where $t = -2\log z/\lambda$ and $\xi_\lambda = (\xi_{\lambda,t} : t \ge 0)$ is a Dawson-Watanabe measure-valued continuous-state branching process, namely the strong solution of the martingale problem associated to the generator

$$\mathcal{L} = \frac{1}{2}\int_{\mathbb{X}} \mu(dx)\frac{\delta^2}{\delta\mu(x)^2} + \frac{1}{2}\theta\int_{\mathbb{X}} \nu(dx)\frac{\delta}{\delta\mu(x)} - \frac{1}{2}\lambda\int_{\mathbb{X}} \mu(dx)\frac{\delta}{\delta\mu(x)}, \tag{19}$$

where $\delta/\delta\mu(x)$ are Gateaux derivatives. The domain $\mathcal{D}(\mathcal{L})$ of the generator is given by the space

$$\left\{\varphi : \varphi(\mu) \equiv F\left(\mu(f_1), \ldots, \mu(f_d)\right), F \in C_c^2(\mathbb{R}_+^d), f_1, \ldots, f_d \in C(\mathbb{X}), d \in \mathbb{N}\right\},$$

where $C(\mathbb{X})$ is the space of all continuous functions on $\mathbb{X}$ and $C_c^2(\mathbb{R}_+^d)$ the space of all continuous, twice differentiable functions on $\mathbb{R}_+^d$ with compact support. We will refer to any such process as a Γ-DW process. The joint Laplace functional transform

$$\phi_t(f,g) = \mathrm{e}^{-c\left[\int_{\mathbb{X}} \log(1+\frac{1}{\lambda}[f(x)+g(x)+C_{-\lambda}(t)f(x)g(x)])\, P_0(dx)\right]} \tag{20}$$

for every $t \geq 0$, where $C_\lambda(t) := (e^{\lambda t/2} - 1)/\lambda$, has been determined in [11]. Indeed one can see that, if $\lambda = 1$, (20) coincides with (17) with $z := 1 - C_{-1}(t)$. From (20), it is immediate to see that the re-scaled DW process $\xi^* = \lambda\xi_\lambda$ is a process whose bivariate distributions (ξ_s^*, ξ_{s+t}^*) have, for every $s, t \geq 0$, canonical correlations $\{z^n\}$ given by

$$z = 1 - \lambda C_{-\lambda}(t) = e^{-\lambda t/2}.$$

In [11] an expansion was derived for the transition function of the Γ-DW process that can be used straight away for all bivariate gamma measures in this class, i.e. induced by $\zeta(x) = z$. One can then state the following result as a direct application of formula (1.8) in [11], where we just substitute $1 - \lambda C_{-\lambda}(t)$ with z.

Corollary 3. *Let $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ be a pair of gamma CRMs with canonical correlations driven by a measure $K = z\widetilde{\mu}$, where $z \in (0,1]$ is a constant and $\widetilde{\mu}$ is a Γ_{cP_0} CRM. The conditional distribution of $\widetilde{\mu}_2$ given $\widetilde{\mu}_1 = m_1$ can be expanded as*

$$p_t(m_1, \mathrm{d}\gamma) = \sum_{n=0}^{\infty} \mathrm{Po}_{(\beta\bar{m}_1)}(n) \int_{\mathbb{X}^n} P_{m_1}^n(\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_n)\, \Gamma_{(c+n)P_n^*, (1+\beta)^{-1}}(\mathrm{d}\gamma) \tag{21}$$

where: $\beta = z/(1-z)$, $\bar{m}_1 = m_1(\mathbb{X})$, $P_{m_1}(A) := m_1(A)/\bar{m}_1$,

$$P_n^*(\mathrm{d}x) = [cP_0(\mathrm{d}x) + n\sum_{i=1}^n \delta_{x_i}(\mathrm{d}x)]/(c+n)$$

and, for every $\beta > 0$, $\Gamma_{\alpha,\beta}(A) = \beta \times \Gamma_{\alpha,1}(A)$, $A \in \mathscr{X}$.

An algorithm for simulating from (the finite-dimensional distributions of) (21) is therefore Algorithm A.3 with $b = \beta$.

5.2. ζ degenerate at a deterministic function

Suppose that $\zeta(x,\omega) = z(x)$ for every $\omega \in \Omega$ and x in $\mathbb{X}$, where $z : \mathbb{X} \to [0,1]$ is some fixed function. Furthermore, set again $G = cP_0$. Then $K(A) =$

$\int_A z(x)\,\widetilde{\mu}(\mathrm{d}x)$ for any A in $\mathscr{X}$. In this case

$$Z_A \stackrel{d}{=} \int_A z(x)\,\widetilde{p}_A(\mathrm{d}x) = \int_0^1 x\;\widetilde{p}_{\tau_A}(\mathrm{d}x)$$

that is, Z_A reduces to a linear functional of a Dirichlet process with *deterministic parameter* τ_A such that $\tau_A(C) = cP_0(A \cap z^{-1}(C))$ for any $C \in \mathscr{B}([0,1])$. Hence, one has a simplification of the canonical correlations. Indeed, if one defines $r_{j,i} = c\int_{A_i} z^j(x)\,P_{0,A_i}(\mathrm{d}x)$, then

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \prod_{i=1}^d \frac{1}{(c)_{n_i}} \sum_{k_i=1}^{n_i} B_{n_i,k_i}(r_{1,i}, 2r_{2,i}, \ldots, (n_i - k_i + 1)!r_{n_i-k_i+1,i})$$

Having ascertained that $\rho_{\boldsymbol{n}}$ has a product form, one finds out that in this case the vectors $(\widetilde{\mu}_1(A_i), \widetilde{\mu}_2(A_i))$, for $i = 1, \ldots, d$, are independent. Hence $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ is a bivariate CRM.

5.3. Random constant ζ

Suppose $G = cP_0$ and $\zeta(x,\omega) = Z(\omega)$ for every $x \in \mathbb{X}$ and ω in Ω, where Z is a random variable taking values in $[0,1]$. Then $K = Z\,\widetilde{\mu}$, where $\widetilde{\mu}$ is still a gamma CRM with parameter measure cP_0. Under this circumstance, one has $Z_A \stackrel{d}{=} Z$ for any A in $\mathscr{X}$ so that by virtue of Theorem 3

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \mathbb{E}\left[Z^{|\boldsymbol{n}|}\right]$$

which does not depend on $\mathcal{A} = \{A_1, \ldots, A_d\}$. Unlike the cases where Z is degenerate at a point z in $[0,1]$, as in Section 5.1, one has $\rho_{\boldsymbol{n}}(\mathcal{A}) \neq \prod_{i=1}^d \rho_{n_i}(\{A_i\})$ and the increments of $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ are not independent. Nonetheless, it is easy to find a useful representation for the conditional distribution of $\widetilde{\mu}_2$ given $\widetilde{\mu}_1$, namely

$$p_G(m_1, \mathrm{d}\gamma) = \mathbb{E}\left[p_B(m_1, \mathrm{d}\gamma)\right]$$

where $B := Z/1 - Z$ and, conditional on $B = \beta$, p_β is as in (21). This also suggests a simple algorithm for generating $\widetilde{\boldsymbol{\mu}}_{i,\mathcal{A}}$ conditional on $\widetilde{\boldsymbol{\mu}}_{j,\mathcal{A}} = \boldsymbol{x} \in \mathbb{R}_+^d$, for $i,j \in \{1,2\} : i \neq j$:

Algorithm A.4.

(0) Initialize by fixing a distribution P_Z on $[0,1]$.

(i) Sample $Z = z$ from P_Z.

(ii) Run Algorithm A.3. initialized by $(z/1-z), \boldsymbol{x}, cP_0(\mathcal{A})$.

A particular choice of P_Z leads to a more explicit formula that is pointed out below.

Corollary 4. *Suppose Z has a beta distribution with parameters $(\eta c, (1-\eta)c)$, where $\eta \in (0,1)$. The canonical correlations are, then, of the form*

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \frac{(\eta c)_{|\boldsymbol{n}|}}{(c)_{|\boldsymbol{n}|}} \tag{22}$$

for any collection $\mathcal{A} = \{A_1, \dots, A_d\}$ of d pairwise disjoint and measurable subsets of $\mathscr{X}$. Moreover, if $f_i : \mathbb{X} \to \mathbb{R}$ are measurable functions such that $\int \log(1+|f|)\mathrm{d}P_0 < \infty$, for any $i = 1, 2$, the Laplace functional transform of $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ evaluated at (f_1, f_2) is

$$\phi(f,g) = \phi_{(cP_0)}(f_1)\,\phi_{(cP_0)}(f_2)\,\mathbb{E}\left[\left(1 + \int \theta\,\mathrm{d}\widetilde{p}_{cP_0}\right)^{-\eta c}\right] \tag{23}$$

where $\widetilde{p}_{cP_0}$ is a Dirichlet process with parameter measure cP_0 and $\theta = f_1 f_2 (1+f_1)^{-1}(1+f_2)^{-1}$.

In (23) one notices that the expectation on the right–hand side is the generalized Cauchy–Stieltjes transform of the mean $\int \theta \mathrm{d}\widetilde{p}_{cP_0}$ of a Dirichlet process. Note that if f_1 and f_2 are simple functions taking on a finite number of values, i.e. $f_1 = \sum_{i=1}^d s_i \mathbb{1}_{A_i}$, $f_2 = \sum_{i=1}^d t_i \mathbb{1}_{A_i}$ for a collection $A_1, \dots A_d$, of pairwise disjoint sets in $\mathscr{X}$, such a transform can be expressed in terms of the fourth Lauricella hypergeometric function F_D. Indeed, upon setting $\theta_i = s_i t_i (1+s_i)^{-1}(1+t_i)^{-1}$ and $\alpha_i = cP_0(A_i)$, for $i = 1, \dots, d$, one has

$$\frac{\phi(f_1,f_2)}{\phi_{(cP_0)}(f_1)\phi_{(cP_0)}(f_2)} = \frac{\prod_{i=1}^d \Gamma(\alpha_i)}{\Gamma(|\boldsymbol{\alpha}|)} F_D\left(\eta c, \alpha_1, \dots, \alpha_d; c; \theta_1, \dots, \theta_d\right)$$

where $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$.

5.4. Model with random elements in common

An alternative construction of vectors of CRMs being in canonical correlation is hinted at in [14], where the authors study a class of Poisson random measures $(\widetilde{J}_1, \widetilde{J}_2)$ such that $\widetilde{J}_i = J_i + J_0$ where J_1, J_2 and J_0 are independent Poisson random measures with $J_1 \stackrel{d}{=} J_2$. Dependence is, thus, induced by a common source of randomness in J_0. A finite-dimensional version of this model was proposed in [17] where the connection with canonical correlations was studied for a large class of marginals including multivariate Poisson and multivariate Gamma. The particular case where J_0 has intensity measure $\eta c P_0$ and J_i, for $i = 1, 2$, have intensity $(1-\eta)cP_0$ ($\eta \in [0,1]$) was employed by [21] to build dependent nonparametric priors for inference on partially exchangeable data. In our measure-valued setting, with Gamma marginals, the latter model can be viewed as the case where $\zeta = 1$ and $G = \eta c P_0$, thus yielding

Corollary 5. *Let μ_1, μ_2 and μ_0 be independent $\Gamma_{(1-\eta)cP_0}$, $\Gamma_{(1-\eta)cP_0}$ and $\Gamma_{\eta cP_0}$ CRMs, respectively. Suppose K is a $\Gamma_{\eta cP_0}$ CRM, with $\eta \in [0,1]$. The corresponding vector $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ of gamma CRMs in canonical correlation is the model with random elements in common*

$$\widetilde{\mu}_1 = \mu_1 + \mu_0 \qquad\qquad \widetilde{\mu}_2 = \mu_2 + \mu_0$$

and the canonical correlations are

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \prod_{i=1}^{d} \frac{(\eta c P_0(A_i))_{n_i}}{(cP_0(A_i))_{n_i}}. \tag{24}$$

for any collection $\mathcal{A} = \{A_1, \ldots, A_d\}$ of disjoint and measurable subsets of $\mathbb{X}$.

Hence, it can be seen that the random variables $Z_{A_1}, \ldots, Z_{A_d}$ that determine $\rho_{\boldsymbol{n}}$ through (12) are independent beta with respective parameters $(\eta c P_0(A_i)$, $(1-\eta)cP_0(A_i))$, for $i = 1, \ldots, d$.

6. Vectors with independent increments

Another interesting point to investigate concerns the case where a vector of canonically gamma CRMs has independent increments, namely for any two disjoint sets A and B in $\mathscr{X}$, the vectors $(\widetilde{\mu}_1(A), \widetilde{\mu}_2(A))$ and $(\widetilde{\mu}_1(B), \widetilde{\mu}_2(B))$ are independent. Indeed, if such a property holds true, it is well–known that $(\tilde{\mu}_1, \tilde{\mu}_2)$ is characterized by the measure cP_0 and by a Lévy intensity ν on $(\mathbb{R}^+)^2$ such that

$$\int_{\mathbb{R}^+} \nu(s_1, s_2) \mathrm{d}s_1 = \frac{\mathrm{e}^{-s_2}}{s_2} \,\mathrm{d}s_2 \qquad \int_{\mathbb{R}^+} \nu(s_1, s_2) \mathrm{d}s_2 = \frac{\mathrm{e}^{-s_1}}{s_1} \,\mathrm{d}s_1.$$

If one further has $\int \|\boldsymbol{s}\| \, \nu(s_1, s_2) \, \mathrm{d}s_1 \, \mathrm{d}s_2 < \infty$ then

$$\mathbb{E}\left[\mathrm{e}^{-\widetilde{\mu}_1(f) - \widetilde{\mu}_2(g)}\right] = \mathrm{e}^{-c \int_{\mathbb{X}\times(\mathbb{R}^+)^2} [1 - \mathrm{e}^{-s_1 f(x) - s_2 g(x)}] \, \nu(s_1,s_2) \, \mathrm{d}s_1 \mathrm{d}s_2 \, P_0(\mathrm{d}x)}$$

One of the examples discussed in Section 5 yields the property of independence we are interested in and it corresponds to a choice if ζ degenerate at a deterministic function. We are now willing to check whether this is also necessary for $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ to be a bivariate CRM. A simple characterization can be deduced from Theorem 3 so that one has

Corollary 6. *The bivariate random measure $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ with gamma CRM marginals has independent increments if and only if K is a CRM on $\mathbb{X}$.*

The proof of the statement is straightforward and can be deduced by directly applying (2) to $f_1 = s_1 \mathbb{I}_A + t_1 \mathbb{I}_B$ and $f_= s_2 \mathbb{I}_A + t_2 \mathbb{I}_B$, for any A, B disjoint. Note that, since $K(A) \stackrel{d}{=} \widetilde{\mu}(A) Z_A$, one has that K is a CRM if and only if the random variables $Z_{A_1}, \ldots, Z_{A_d}$ are independent for any collection $\mathcal{A} = \{A_1, \ldots, A_d\}$

of pairwise disjoint sets in $\mathscr{X}$ and $d \geq 1$. It then follows that $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ has independent increments if and only if

$$\rho_{\boldsymbol{n}}(\mathcal{A}) = \prod_{i=1}^{d} \rho_{n_i}(A_i)$$

This fact leads us to a characterization of those sequences $\mathscr{Z} = \{Z_A : A \in \mathscr{X}\}$ whose moments give rise to bivariate gamma measures in canonical correlation, with jointly independent increments.

Theorem 5. *A pair $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ of gamma CRMs in canonical correlation is itself a CRM vector if and only if, for every $A \in \mathbb{X}$, Z_A is equal in distribution to the mean of a Dirichlet process with deterministic base measure τ_A.*

This result, besides its relevance for the goals set forth in this paper, is interesting in that it reproduces and completes a result on one-dimensional gamma canonically correlated random variables with infinitely divisible joint distribution, obtained by Griffiths in [13]. We restate Griffiths' result as follows.

Theorem 6 (Griffiths (1970))**.** *The sequence $(\rho_n)_{n\geq 1}$ defines canonical correlations for an infinitely divisible vector $(X, Y) \in \mathbb{R}_+^2$ with marginally $\Gamma_{c,1}$ distributed random variables if and only if $\rho_n = \mathbb{E}\left[(\widetilde{p}(f))^n\right]$, with $\widetilde{p}$ being a Dirichlet process with parameter measure cP_0 and for some measurable function $f : \mathbb{X} \to [0,1]$ such that $\int \log(1 + |f|)\, \mathrm{d}P_0 < \infty$.*

The proof is obvious from Theorem 5 and is omitted.

7. Measure–valued Markov processes with canonical correlations

In this Section we apply Theorem 3 to characterize the class of time-homogeneous, reversible, Feller transition functions $\{P_t : t \geq 0\}$ defined on $M_{\mathbb{X}} \times \mathscr{M}_{\mathbb{X}}$, with Γ_{cP_0} stationary measure such that, for every $\mu \in M_{\mathbb{X}}$ and $t \geq 0$, all the finite dimensional distributions associated to $P_t(\mu, \cdot)$ have multivariate Laguerre polynomial eigenfunctions. In other words, for each such kernel, the corresponding measure-valued Markov process $\{\xi_t : t \geq 0\}$ will be one with canonical autocorrelations. We prove that all such Feller transition functions correspond to the law of a Dawson-Watanabe process with generator (19), time-changed by means of an independent, *one-dimensional* subordinator. From now on we will assume, for simplicity, that $\mathbb{X}$ is a compact metric space. Then $M_{\mathbb{X}}$, endowed with the weak topology, is a locally compact metric space which can be compactified via routine methods such as those indicated in ([5], Section 3.1).

We will derive our characterization in two steps. First, we fix a collection of say d disjoint Borel sets $\mathcal{A} = \{A_1, \ldots, A_d\}$ and correspondingly set $\boldsymbol{X}_t^{\mathcal{A}} := (\xi_t(A_1), \ldots, \xi_t(A_d))$. We will establish necessary and sufficient conditions for $\{\boldsymbol{X}_t^{\mathcal{A}} : t \geq 0\}$ to be a $\mathbb{R}_+^d$-valued Markov process which, for ant $t \geq 0$, satisfies

$$\mathbb{E}\left[\widetilde{L}_{\boldsymbol{\alpha},\boldsymbol{n}}(\boldsymbol{X}_0^{\mathcal{A}})\widetilde{L}_{\boldsymbol{\alpha},\boldsymbol{m}}(\boldsymbol{X}_t^{\mathcal{A}})\right] = \delta_{\boldsymbol{mn}}\, \rho_{\boldsymbol{n}}(\mathcal{A}, t) \qquad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}_+^d \tag{25}$$

for some sequence of functions $\{\rho_{\boldsymbol{n}}(\mathcal{A},t) : \boldsymbol{n} \in \mathbb{Z}_+^d, t \geq 0\}$. As a second step, we will need to guarantee that both markovianity and (25) hold consistently when we let d and $A_1, \ldots, A_d \in \mathscr{X}^d$ vary. It is just such consistency that will lead us to the subordinated Dawson-Watanabe interpretation.

As for the first step, necessary and sufficient conditions for (25) to hold true for any $t \geq 0$ are given in Theorem 2. Hence, we only need to identify conditions on the map $t \mapsto \rho_{\boldsymbol{n}}(\mathcal{A},t)$ that imply the Markov property. Theorem 3 is all we need to study (step 2) the existence of a Markov operator mapping polynomials to polynomials of the same degree for all the finite-dimensional distributions of the process.

7.1. Step 1. Markov gamma vectors

The approach we use is similar to the method suggested in Bochner[3] to study Gegenbauer processes on $[-1,1]$, and employed by Griffiths [12] to characterize one-dimensional Laguerre stochastic processes. Assume $\{\boldsymbol{X}_t^{\mathcal{A}} : t \geq 0\}$ is a time-homogeneous Markov process with canonical autocorrelation and with $\Gamma_{\boldsymbol{\alpha},1} = \times_{i=1}^d \Gamma_{\alpha_i,1}$ as stationary measure, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and $\alpha_i = cP_0(A_i)$. Consider the associated conditional expectation operator

$$P_t f(\boldsymbol{x}) = \mathbb{E}\left[f(\boldsymbol{X}_t^{\mathcal{A}}) \mid \boldsymbol{X}_0^{\mathcal{A}} = \boldsymbol{x}\right] \qquad t \geq 0$$

for every bounded measurable functions $f : \mathbb{R}_+^d \to \mathbb{R}$. Note that, in order for P_t to correspond to a Markov transition kernel, $t \mapsto P_t$ must be continuous in t and P_0 is the identity operator. From (4) one has

$$P_t \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) = \rho_{\boldsymbol{n}}(\mathcal{A},t)\, \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) \qquad \forall t \geq 0.$$

The conditions above and the Chapman-Kolmogorov property $P_{t+s} = P_t P_s$, for all non negative s and t, are satisfied if and only if, for every $\boldsymbol{n} \in \mathbb{Z}_+^d$,

$$\rho_{\boldsymbol{n}}(\mathcal{A},t+s) = \rho_{\boldsymbol{n}}(\mathcal{A},t)\,\rho_{\boldsymbol{n}}(\mathcal{A},s) \qquad \forall s,t \geq 0 \tag{26}$$

$$\rho_{\boldsymbol{n}}(\mathcal{A},0) = 1 \tag{27}$$

$$t \mapsto \rho_{\boldsymbol{n}}(\mathcal{A},t) \qquad \text{continuous} \tag{28}$$

These properties define an ordinary kinetic equation, that is: (26)-(28) hold if and only if there exists a sequence of (non-negative) reals $\lambda_{\boldsymbol{n}}(\mathcal{A})$ such that

$$\rho_{\boldsymbol{n}}(\mathcal{A},t) = \mathrm{e}^{-t\lambda_{\boldsymbol{n}}(\mathcal{A})} \qquad \forall t \geq 0.$$

If the conditions above are satisfied, then $\{P_t : t \geq 0\}$ is a Feller semigroup because it maps the space $L_2(\mathbb{R}_+^d, \Gamma_{\boldsymbol{\alpha},1})$ onto itself (being the Laguerre polynomials an orthogonal basis for it) and, in particular, it maps the space of all bounded continuous functions vanishing at infinity onto itself.

In the light of this discussion, we aim at characterizing multi-indexed sequences $\{\lambda_{\boldsymbol{n}}(\mathcal{A}) : \boldsymbol{n} \in \mathbb{Z}_+^d\}$ that may serve as "rates" for the canonical autocorrelations of $\boldsymbol{X}^{\mathcal{A}}$. It turns out that each such sequence of rates is given by the Laplace exponent, evaluated at $\boldsymbol{n} \in \mathbb{Z}_+^d$, of some multivariate subordinator. The Lévy-Kintchine representation of the Laplace transform of a multivariate subordinator $\boldsymbol{Y} = \{\boldsymbol{Y}_t : t \geq 0\}$, with $\boldsymbol{Y}_t = (Y_{1,t}, \ldots, Y_{d,t})$ for any $t \geq 0$, is ([2], Theorem 3.1),

$$-\frac{1}{t} \log \mathbb{E}\left[e^{-\sum_{i=1}^d \phi_i Y_{i,t}}\right] = \sum_{i=1}^d \phi_i c_i + \int_{\mathbb{R}_+^d} \left(1 - e^{-\sum_{i=1}^d \phi_i u_i}\right) \eta(\mathrm{d}\boldsymbol{u}) \tag{29}$$

for any $(\phi_1, \ldots, \phi_d) \in \mathbb{R}_+^d$, from some constants $c_i \geq 0,\ i = 1, \ldots, d$ and Lévy measure η on $\mathbb{R}_+^d$. Note that η must be such that $\int_{\mathbb{R}_+^d} (\|\boldsymbol{u}\| \wedge 1)\eta(\mathrm{d}\boldsymbol{u}) < \infty$, with $\|\boldsymbol{u}\|$ denoting the Euclidean norm. This implies that $\boldsymbol{Y}_t$ has, for every $t \geq 0$, a multivariate infinitely divisible law. Henceforth, we shall denote the *Laplace exponent* of $\boldsymbol{Y}$, i.e. the right–hand side of (29), as ψ. A consequence of (26)-(28) is the following

Theorem 7. *For every $d \geq 1$ and every collection $\mathcal{A} = (A_1, \ldots, A_d) \in \mathscr{X}^*$, a time-homogeneous Markov process $\{\boldsymbol{X}_t^{\mathcal{A}} : t \geq 0\}$ in $\mathbb{R}_+^d$, with $\Gamma_{\boldsymbol{\alpha},1}$ stationary distribution, is time-reversible with canonical autocorrelation function ρ if and only if there is a multivariate subordinator $\boldsymbol{Y}_{\mathcal{A}} = \{\boldsymbol{Y}_t^{\mathcal{A}} : t \geq 0\}$ with corresponding Lévy exponent $\psi_{\mathcal{A}}$ such that, for every $\boldsymbol{n} \in \mathbb{Z}_+^d$, and $\boldsymbol{x} \in \mathbb{R}_+^d$,*

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \mathrm{e}^{-t\psi_{\mathcal{A}}(\boldsymbol{n})}. \tag{30}$$

7.2. Step 2. Markov gamma measure-valued processes

We are now ready to derive the characterization of measure-valued gamma processes we are seeking, as Dawson-Watanabe processes time-changed by a one-dimensional subordinator. From Section 5.1, we know that a DW process $\xi = \{\xi_t : t \geq 0\}$ with criticality parameter $\lambda = 1$ has canonical correlations

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \mathrm{e}^{-|\boldsymbol{n}|t/2}$$

for every $\boldsymbol{n} \in \mathbb{Z}_+^d$, $d \in \mathbb{N}$, $t \geq 0$ and $\mathcal{A} = \{A_1, \ldots, A_d\}$. The exponent does not depend on A. Moreover, it corresponds to the Laplace exponent of a pure drift one-dimensional subordinator and, thus, all its finite-dimensional distributions on disjoint subsets are of the form (30). Now take any (one-dimensional) subordinator $S = \{S_t : t \geq 0\}$, driven by a Lévy measure ν, defined on the same probability space as and independent of ξ. Denote with ψ_ν its Lévy exponent and define the process ξ^S as

$$\xi^S := \left\{\xi_t^S : t \geq 0\right\} \overset{d}{=} \{\xi_{S_t} : t \geq 0\}$$

The process ξ^S is still a Markov process, since every time-change via a subordinator maps any Markov process to a new Markov process (see e.g. [22], Chapter 6). The stationary measure is unchanged so that, if $\{P_t^{\mathrm{DW}} : t \geq 0\}$ and $\{P_t^S : t \geq 0\}$ denote the semigroup corresponding to the transition function of

ξ and ξ^S respectively, then, for every bounded test function $\varphi = \varphi(\mu)$ in the domain $\mathcal{D}(\mathcal{L})$ of the generator of $(P_t^{\mathrm{DW}} : t \geq 0)$,

$$\mathbb{E}\left[P_t^S \varphi(\mu)\right] = \mathbb{E}\left[\mathbb{E}_{cP_o}\left[P_{S_t}^{\mathrm{DW}} \varphi(\mu)\right] \mid S_t\right] = \mathbb{E}_{cP_0} \varphi(\mu),$$

where $\mathbb{E}_{cP_0}$ indicated the expectation with respect to the DW stationary measure Γ_{cP_0}. It can be seen that the process of the finite-dimensional distributions of ξ^S obeys to

$$P_t^S \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) = \mathbb{E}\left[\mathrm{e}^{-|\boldsymbol{n}|S_t/2}\right] \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) = \mathrm{e}^{-t\psi_\nu(|\boldsymbol{n}|/2)} \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x})$$

for every $\boldsymbol{n} \in \mathbb{Z}_+^d$ and $\mathcal{A} \in \mathscr{X}^*$ such that $cP_0(\mathcal{A}) = \boldsymbol{\alpha}$. Consequently ξ^S has Markov canonical correlation coefficients satisfying (30) and, thus, subordinated Γ-DW processes have canonical auto-correlation. Moreover, they are Feller processes since ξ is a Feller process. We have just proved the sufficiency part of the next Theorem. A less immediate task is to prove the converse.

Theorem 8. *A time-reversible, time-homogeneous Feller process $\xi = \{\xi_t : t \geq 0\}$ with values in $(M_{\mathbb{X}}, \mathscr{M}_{\mathbb{X}})$ and stationary measure Γ_{cP_0}, has canonical auto-correlations if and only if ξ can be obtained as a Γ-DW process with the same stationary distribution and time-changed via an independent one-dimensional subordinator. The generator of ξ is thus given by*

$$\mathcal{L}^S \varphi(\mu) = \mathcal{L}\varphi(\mu) + \int_0^\infty P_s^{DW} \varphi(\mu)\ \nu(\mathrm{d}s)$$

where $\mathcal{L}$ is the generator of the Γ-DW diffusion with semigroup $(P_t^{DW} : t \geq 0)$ (see Section 5.3) and ν is the Lévy measure of the subordinator. $\mathcal{D}(\mathcal{L})$ is a core for the domain of $\mathcal{L}^S$.

7.3. Algorithms and further properties

The system of finite–dimensional distributions of a subordinated DW process ξ^S is such that the distribution of (ξ_s^S, ξ_{s+t}^S) is a member of the class examined in Section 5.3, for fixed $s, t \geq 0$. In that class, the dependence between two canonically correlated gamma measure was described by a random measure $K \overset{d}{=} Z\,\widetilde{\mu}$, where $\widetilde{\mu}$ is a $\Gamma_{\theta P_0}$ CRM and Z is a random variable independent of $\widetilde{\mu}$ and whose distribution does not depend on $\mathcal{A}$. For subordinated DW processes this property can be rephrased by saying that ξ_s^S and ξ_{s+t}^S are canonically correlated and their dependence is determined by a random measure K_t such that

$$K_t \overset{d}{=} e^{-S_t} \widetilde{\mu}.$$

Here we stress two important consequences. The first one is about an algorithm for simulation from the transition function of $\boldsymbol{X}^{\mathcal{A}}(S)$.

Corollary 7. *Let $\{\xi_t : t \geq 0\}$ be a measure-valued reversible Markov process with canonical autocorrelations and Γ_{cP_0} stationary distribution. There exists a $\mathbb{R}_+$-valued subordinator $S = \{S_t : t \geq 0\}$ such that, for every $t \geq 0$, Algorithm A.4 initialized by the probability distribution of e^{-S_t}, generates a realization of $\{(\xi_0(A_i), \xi_t(A_i)) : i = 1, \ldots, d\}$ for every d and every measurable partition $A = (A_1, \ldots, A_d)$ of $\mathbb{X}$.*

The second consequence relates to the findings of Theorem 5. In particular, suppose S is a degenerate process such that, for every t, $S_t = 2ct$ for some positive constant $2c$. If ξ is a Γ-DW process with criticality $\lambda = 1$, the canonical autocorrelations of $\{\xi_t^S : t \geq 0\}$ are determined by powers of $z = e^{-ct}$. We have seen in Section 5.3 that this forms the canonical correlation sequence of a re-scaled DW process with criticality parameter $\lambda = 2c$. These are the only instances of Markov canonical autocorrelations that can be represented via moments of a degenerate random variable $z \in [0, 1]$. By Theorem 5, this just proves a new characterization of the DW process.

Theorem 9. *Let $\{\xi_t^* : t \geq 0\}$ be a measure-valued Markov process with Γ_{cP_0} stationary distribution and canonical auto-correlations. The pair (ξ_0^*, ξ_t^*) is a bivariate completely random measure, for every t, if and only if it is a DW measure-valued branching diffusion with immigration, up to a deterministic re-scaling, namely $\{\xi_t^* : t \geq 0\} = \{\lambda \xi_\lambda(t) : t \geq 0\}$ for some $\lambda \in \mathbb{R}$.*

Appendix A: Proof of Lemma 1

From the algorithm, one can see that, for every $x \in [0, \infty)$, the conditional distribution of Y given X has the representation

$$p_b(x, dy) = \sum_{n=0}^{\infty} \frac{(bx)^n e^{-bx}}{n!} \Gamma_{\alpha+n,(1+b)^{-1}}(dy). \tag{31}$$

Thus the conditional Laplace transform is:

$$\begin{aligned}
\mathbb{E}\left[e^{-tY} \mid X = x\right] &= \sum_{n=0}^{\infty} \frac{(bx)^n e^{-bx}}{n!} \phi_{\alpha+n,(1+b)^{-1}}(t) \\
&= \left(1 + \frac{t}{1+b}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(bx)^n e^{-bx}}{n!} \left(\frac{1+b}{1+b+t}\right)^n \\
&= \left(1 + \frac{t}{1+b}\right)^{-\alpha} \exp\left\{bx\left(\frac{1+b}{1+b+t} - 1\right)\right\} \\
&= (1 + t(1-z))^{-\alpha} e^{-\frac{xtz}{1+t(1-z)}},
\end{aligned} \tag{32}$$

where the second equality comes from the form of the gamma Laplace transform, the third from the form of the Poisson probability generating function, and the last one by substituting $z = b/(1+b)$. The right-hand side of (32) is equal to (10) and that suffices to prove both the necessity and the sufficiency part of the Lemma. $\square$

Appendix B: Proof of Theorem 2.

For any $d \in \mathbb{N}$, conditioning on $\boldsymbol{B} = (b_1, \dots, b_d)$ the distribution is the same one would obtain by running d independent copies of algorithm A.1, each i-th copy being initialized by α_i, b_i, $i = 1, \dots, d$. Then from Lemma 1

$$\begin{aligned}\mathbb{E}\left[\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{X})\,\widetilde{L}_{\boldsymbol{m},\boldsymbol{\alpha}}(\boldsymbol{Y})\right] &= \mathbb{E}\left[\mathbb{E}\left[\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{X})\,\widetilde{L}_{\boldsymbol{m},\boldsymbol{\alpha}}(\boldsymbol{Y}) \mid \boldsymbol{B}\right]\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{d} \delta_{n_i m_i}\left(\frac{B_i}{1+B_i}\right)^{n_i}\right] \\ &= \delta_{\boldsymbol{n}\boldsymbol{m}}\mathbb{E}\left[\prod_{i=1}^{d} Z_i^{n_i}\right]\end{aligned}$$

where $Z_i := B_i/(1-B_i),\ i = 1, \dots, d$ and the sufficiency part follows.
For the necessity, we use an approach similar to one used in [14] to construct bivariate Poisson vectors. Let us suppose the pair $(\boldsymbol{X}, \boldsymbol{Y})$ have canonical correlation sequence $\rho(\boldsymbol{\alpha}) = \{\rho_{\boldsymbol{n}}(\boldsymbol{\alpha})\}$. Let f be any function on $\mathbb{R}_+^d$ with finite $\times_{i=1}^d \Gamma_{\alpha_i,1}$-variance and set

$$\hat{f}(\boldsymbol{n}) := \mathbb{E}_{\boldsymbol{\alpha}}\left[f(\boldsymbol{X})\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x})\right].$$

From (4),

$$\mathbb{E}\left[f(\boldsymbol{Y}) \mid \boldsymbol{X} = \boldsymbol{x}\right] = \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \frac{\hat{f}(\boldsymbol{n})}{c_{\boldsymbol{n},\boldsymbol{\alpha}}}\rho_{\boldsymbol{n}}(\boldsymbol{\alpha})\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}).$$

Then for every $\boldsymbol{s} \in \mathbb{R}_+^d$,

$$\begin{aligned}\mathbb{E}\left[e^{-\langle \boldsymbol{s},\boldsymbol{Y}\rangle} \mid \boldsymbol{X} = \boldsymbol{x}\right] &= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \rho_{\boldsymbol{n}}(\boldsymbol{\alpha})\,\frac{1}{c_{\boldsymbol{n},\boldsymbol{\alpha}}}\prod_{i=1}^{d}\widetilde{L}_{n_i,\alpha_i}(x_i)\,\psi_{n_i,\alpha_i}(s_i) \\ &= \left\{\prod_{i=1}^{d}(1+s_i)^{-\alpha_i}\right\}\sum_{\boldsymbol{n}\in\mathbb{Z}_+^d}\rho_{\boldsymbol{n}}(\boldsymbol{\alpha})\prod_{i=1}^{d}\frac{\widetilde{L}_{n_i,\alpha_i}(x_i)}{n_i!}\,\frac{(-s_i)^{n_i}}{(s_i+1)^{n_i}}.\end{aligned} \tag{33}$$

Note that, replacing each $\rho_{\boldsymbol{n}}(\boldsymbol{\alpha})$ with 1, the series becomes

$$\begin{aligned}\sum_{\boldsymbol{n}\in\mathbb{Z}_+^d}\prod_{i=1}^{d}\frac{\widetilde{L}_{n_i,\alpha_i}(x_i)}{n_i!}\,\frac{(-s_i)^{n_i}}{(s_i+1)^{n_i}} &= \prod_{i=1}^{d} G\left(\frac{-s_i}{s_i+1}, x_i\right) \\ &= \prod_{i=1}^{d}\left(\frac{1}{1+s_i}\right)^{-\alpha_i} e^{-x_i s_i},\end{aligned} \tag{34}$$

where $G(r, x)$ is the generating function (6). Since

$$\rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) = \prod_{i=1}^{d} c_{n_i}\,\mathbb{E}\left[\prod_{i=1}^{d}\widetilde{L}_{n_i,\alpha_i}(X_i)\widetilde{L}_{n_i,\alpha_i}(Y_i)\right]$$

by the Cauchy-Schwartz inequality

$$|\rho_{\boldsymbol{n}}(\boldsymbol{\alpha})|^2 \leq \prod_{i=1}^{d} c_{n_i}^2 \, \mathbb{E}\left[\widetilde{L}^2_{n_i,\alpha_i}(X_i)\right] \, \mathbb{E}\left[\widetilde{L}^2_{n_i,\alpha_i}(Y_i)\right] = 1.$$

Then $0 \leq \rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) \leq 1$. Hence (34) implies, in particular, that the sum in (33) converges absolutely. Now, for every $x > 0$,

$$\mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{d} s_i \frac{Y_i}{X_i}} \mid X_i = x, \, i = 1, \ldots, d\right] =$$

$$\prod_{i=1}^{d}\left(1+\frac{s_i}{x}\right)^{-\alpha_i} \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) \prod_{i=1}^{d} \frac{\widetilde{L}_{n_i,\alpha_i}(x)}{n_i!} \frac{(-s_i)^{n_i}}{(s_i+x)^{n_i}} =: \phi_{\rho,x}(\boldsymbol{s}).$$

The limit of $\phi_{\rho,x}(\boldsymbol{s})$, as $x \to \infty$, is still the Laplace transform of a d-dimensional probability distribution, again thanks to the boundedness of the canonical correlation coefficients. In particular,

$$\phi_\rho(\mathbf{s}) := \lim_{x\to\infty} \phi_{\rho,x}(\boldsymbol{s}) = \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) \prod_{i=1}^{d} \frac{(-s_i)^{n_i}}{n_i!}$$

is continuous at the origin. Thus $\{\rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) : \boldsymbol{n} \in \mathbb{Z}_+^d\}$ has the interpretation as the moment sequence of a random variable $\boldsymbol{Z} \in \mathbb{R}_+^d$. Since one further has $\rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) \leq 1$, it is the moment sequence of a random variable in $[0,1]^d$.
Finally, it remains to prove that every canonical correlation sequence correspond to algorithm of the type A.2. Let f be any function on $\mathbb{R}_+^d$ with finite $\times_{i=1}^{d}\Gamma_{\alpha_i,1}$-variance. Then

$$\begin{aligned}
\mathbb{E}\left[f(\boldsymbol{Y}) \mid \boldsymbol{X} = \boldsymbol{x}\right] &= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \frac{\hat{f}(\boldsymbol{n})}{c_{\boldsymbol{n},\boldsymbol{\alpha}}} \rho_{\boldsymbol{n}}(\boldsymbol{\alpha}) \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \frac{\hat{f}(\boldsymbol{n})}{c_{\boldsymbol{n},\boldsymbol{\alpha}}} \mathbb{E}\left[\boldsymbol{Z}^{\boldsymbol{n}}\right] \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}) \\
&= \mathbb{E}\left[\sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \frac{\hat{f}(\boldsymbol{n})}{c_{\boldsymbol{n},\boldsymbol{\alpha}}} \boldsymbol{Z}^{\boldsymbol{n}} \widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x})\right],
\end{aligned}$$

where $\boldsymbol{Z}^{\boldsymbol{n}} := \prod_{i=1}^{d} Z_i^{n_i}$. The sums can be interchanged because $\sum_{\boldsymbol{n}} {c_{\boldsymbol{n},\boldsymbol{\alpha}}}^{-1}(\hat{f}(\boldsymbol{n}))^2 < \infty$ and both $|\rho_{\boldsymbol{n}}| \leq 1$ and $|\boldsymbol{Z}^{\boldsymbol{n}}| \leq 1$, thus all the sums converge. The inner series can therefore be interpreted as a version of the conditional expectation

$$\mathbb{E}\left[f(\boldsymbol{Y}) \mid \boldsymbol{X} = \boldsymbol{x}, \boldsymbol{Z} = \boldsymbol{z}\right]$$

where $(\boldsymbol{X}, \boldsymbol{Y})$ is the pair generated by step (iii) of Algorithm A.2, conditional on the realization $\boldsymbol{B} = (z_i/(1-z_i) : i = 1, \ldots, d)$. The proof is thus complete. $\square$

Appendix C: Proof of Theorem 3

Suppose (i) holds true and introduce simple functions $f = \sum_{i=1}^{d} s_i \mathbb{1}_{A_i}$ and $g = \sum_{i=1}^{d} t_i \mathbb{1}_{A_i}$, where $A_1, \ldots, A_d$ are measurable disjoint subsets of $\mathbb{X}$. From Theorem 2, one then has

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-\widetilde{\mu}_1(f)-\widetilde{\mu}_2(g)}\right] &= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \rho_{\boldsymbol{n}}(\mathcal{A}) \prod_{i=1}^{d} \frac{\psi_{n_i,\alpha_i}(s_i)\,\psi_{n_i,\alpha_i}(t_i)}{(\alpha_i)_{n_i}\, n_i!} \\ &= \left\{\prod_{i=1}^{d} \phi_{\alpha_i,1}(s_i)\,\phi_{\alpha_i,1}(t_i)\right\} \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \rho_{\boldsymbol{n}}(\mathcal{A}) \prod_{i=1}^{d} \frac{(\alpha_i)_{n_i}}{n_i!}(\theta_i)^{n_i} \\ &= \left\{\prod_{i=1}^{d} \phi_{\alpha_i,1}(s_i)\,\phi_{\alpha_i,1}(t_i)\right\} \mathbb{E}\left[\prod_{i=1}^{d}\sum_{n_i=0}^{\infty} \frac{(Y_i Z_i)^{n_i}}{n_i!}\,\theta^{n_i}\right] \qquad (35) \end{aligned}$$

for some random vector $(Z_1, \ldots, Z_d) \in [0,1]^d$, where $\theta_i = s_i t_i/[(1+s_i)(1+t_i)]$, for each $i = 1, \ldots, d$, and the Y_i's are independents random variables with respective distributions $\Gamma_{\alpha_i,1}$. The series expansion in the expected value above leads to

$$\mathbb{E}\left[\mathrm{e}^{-\widetilde{\mu}_1(f)-\widetilde{\mu}_2(g)}\right] = \left\{\prod_{i=1}^{d} \phi_{\alpha_i,1}(s_i)\,\phi_{\alpha_i,1}(t_i)\right\} \int_{(\mathbb{R}^+)^d} \mathrm{e}^{-\langle\boldsymbol{\theta},\boldsymbol{v}\rangle}\, p_{\mathcal{A}}(\mathrm{d}\boldsymbol{v}) \qquad (36)$$

where $p_{\mathcal{A}}$ is a probability distribution on $(\mathbb{R}^+)^d$ such that

$$\int_{(\mathbb{R}^+)^d} \boldsymbol{v}^{\boldsymbol{n}}\, p_{\mathcal{A}}(\mathrm{d}\boldsymbol{v}) = \mathbb{E}\left[\prod_{i=1}^{d} Z_i^{n_i}\right] \prod_{i=1}^{d} (\alpha_i)_{n_i}$$

and $\boldsymbol{v}^{\boldsymbol{n}} = \prod_{i=1}^{d} v_i^{n_i}$. Our next goal consists in showing that the collection $\mathscr{P}' := \{p_{\mathcal{A}} : \mathcal{A} \in \mathscr{X}^d,\ d \ge 1\}$ uniquely identifies a random measure on $(\mathbb{X}, \mathscr{X})$. To this end, note that the system $\mathscr{P}'$ can be extended to define a family $\mathscr{P} = \{p_{B_1,\ldots,B_d} : B_i \in \mathscr{X},\ d \ge 1\}$ in such a way that $\mathscr{P}$ is consistent and

$$p_{\mathscr{B}}(\{(x,y,z) \in (\mathbb{R}^+)^3 :\ x + y = z\}) = 1$$

for any A and B in $\mathscr{X}$ such that $A \cap B = \varnothing$ and $\mathscr{B} = \{A, B, A\cup B\}$. Moreover, if $d = 1$ and $\mathcal{A} = \{A_1\}$, note that $p_{\mathcal{A}}$ coincides with the probability distribution of $Z_{A_1}\,\widetilde{\mu}(A_1)$, where $\widetilde{\mu}$ is a Γ_{cP_0} CRM. Hence, if $(B_n)_{n\ge1}$ is a sequence of elements in $\mathscr{X}$ such that $\lim_n B_n = \varnothing$ and $\mathscr{B}_n = \{B_n\}$ for any $n \ge 1$, $p_{\mathscr{B}_n} \Rightarrow \delta_0$ almost surely, where $\Rightarrow$ stands for weak convergence. By Theorem 5.4 in [15] there exists a unique random measure, say K, on $(\mathbb{X}, \mathscr{X})$ admitting $\mathscr{P}$ as its system of finite–dimensional distributions. Moreover, since $K(A) \stackrel{d}{=} Z_A\,\widetilde{\mu}(A)$ for any $A \in \mathscr{X}$, then $\mathbb{P}[K(A) > x] \le \mathbb{P}[\widetilde{\mu}(A) > x]$ for any A in $\mathscr{X}$ and $x \in \mathbb{R}^+$.

Let us, now, assume that (ii) holds true. With $f = \sum_{i=1}^{d} s_i \mathbb{1}_{A_i}$ and $g = \sum_{i=1}^{d} t_i \mathbb{1}_{A_i}$ and the d sets $A_1, \dots, A_d$ in $\mathscr{X}$ being pairwise disjoint, the Laplace transform $\phi(f,g)$ in (3) is of the form (36), and since $\widetilde{\mu}$ dominates stochastically K, $K(A) \stackrel{d}{=} Z_A \widetilde{\mu}(A)$ for any A in $\mathscr{X}$, with Z_A and $\widetilde{\mu}(A)$ independent. Thus an expansion for $\phi(f,g)$ is precisely of the form (35), which shows that $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ are in canonical correlations with $\rho_{\boldsymbol{n}}$ as in (12). Moreover, By additivity of random measures, for A and B disjoint, $K(A \cup B) = K(A) + K(B)$. This entails, being $\widetilde{\mu}(A)$ and $\widetilde{\mu}(B)$ independent Gamma random variables,

$$Z_{A \cup B} \stackrel{d}{=} \frac{Z_A \widetilde{\mu}(A) + Z_B \widetilde{\mu}(B)}{\widetilde{\mu}(A \cup B)}$$

(it can be easily seen by taking the moments of any order of both sides) and (13) follows thus completing the proof. □

Appendix D: Proof of Corollary 1

Let $\{A_0, A_1\}$ be a measurable partition of $A \in \mathscr{X}$. By construction,

$$\mathbb{E}\,[Z_A^n] = \frac{(G(A))_n}{(cP_0(A))_n} \mathbb{E}\,[V_A^n] \qquad A \in \mathscr{X}. \tag{37}$$

Assume $\mathscr{Z}$ satisfies (13). Then

$$\mathbb{E}\,[Z_A^n] = \frac{(G(A))_n}{(cP_0(A))_n} \sum_{j=0}^{n} \binom{n}{j} \frac{(G(A_0))_j (G(A_1))_{n-j}}{(G(A))_n} \mathbb{E}\left[V_{A_0}^{j} V_{A_1}^{n-j}\right],$$

which, by virtue of (37), implies that $\mathscr{V}$ satisfies (13) with respect to G. The proof of the converse proceeds along similar lines. □

Appendix E: Proof of Theorem 4

Suppose, first, that (15) and (i)–(ii) hold true. Hence, for any pair A and B of disjoint sets in $\mathscr{X}$, one has

$$Z_{A \cup B} \stackrel{d}{=} \int_{[0,1]} x\, \widetilde{p}_{\tau_A + \tau_B}(\mathrm{d}x).$$

From (ii) and a well–known property of Dirichlet processes, conditional on τ_A and τ_B one has

$$\int_{[0,1]} x\, \widetilde{p}_{\tau_A+\tau_B}(\mathrm{d}x) \stackrel{d}{=} \varepsilon \int_{[0,1]} x\, \widetilde{p}_{\tau_A}(\mathrm{d}x) + (1-\varepsilon) \int_{[0,1]} x\, \widetilde{p}_{\tau_B}(\mathrm{d}x)$$

Hence, one deduces (13) which implies that $\mathscr{Z} = \{Z_A : A \in \mathscr{X}\}$ defines a collection of canonical correlations for a vector of Γ_H CRMs according to (16).

Conversely, suppose that $\mathscr{Z} = \{Z_A : A \in \mathscr{X}\}$ satisfies (13). Let $c > 0$ and $(\widetilde{P}_{c,j})_{j\geq 1}$ denote a random point in the infinite simplex with a so–called GEM(c) distribution, namely the sequence $(B_j)_{j\geq 1}$ defined by $B_j = \widetilde{P}_{c,j}/(1 - \sum_{i=1}^{j-1} \widetilde{P}_{c,i})$ is a sequence of independent and identically distributed (iid) beta random variables with parameters $(1, c)$. With H denoting a finite and non–null measure on $\mathbb{X}$, such that $H(\mathbb{X}) = c$, and $\widetilde{p}_H$ a Dirichlet process on $(\mathbb{X}, \mathscr{X})$ with baseline measure H, it is well–known that

$$\widetilde{p}_H \overset{d}{=} \sum_{j=1}^{\infty} \widetilde{P}_{c,j} \delta_{Y_j} \tag{38}$$

where $(Y_j)_{j\geq 1}$ is a collection of iid random variables independent of $(\widetilde{P}_{c,j})_{j\geq 1}$ and whose probability distribution is H/c. Hence,

$$\int_{\mathbb{X}} x\, \widetilde{p}_H(\mathrm{d}x) = \sum_{j=1}^{\infty} \widetilde{P}_{c,j} Y_j. \tag{39}$$

Set, now, $H = cP_0$ and denote by $\Pi := (\Pi_m)_{m\geq 1}$ an arbitrary sequence of nested and measurable partitions of A such that $\Pi_m := \{A_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \{0,1\}^m\}$. This means that for $\boldsymbol{\varepsilon} \in \{0,1\}^m$, one has $A_{\boldsymbol{\varepsilon}} = A_{\boldsymbol{\varepsilon}0} \cup A_{\boldsymbol{\varepsilon}1}$. From (13) one deduces

$$Z_A \overset{d}{=} \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^m} Z_{A_{\boldsymbol{\varepsilon}}}\, \tilde{p}_A(A_{\boldsymbol{\varepsilon}}) \tag{40}$$

where $\tilde{p}_A$ is a Dirichlet process whose parameter measure H_A is the measure H restricted to set A and such that $H_A(A) = H(A) = cP_0(A)$. Introduce simple functions $\zeta_A^{(m)}(y) = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^m} Z_{A_{\boldsymbol{\varepsilon}}} \delta_y(A_{\boldsymbol{\varepsilon}})$ and note that for any $m \geq 1$

$$Z_A \overset{d}{=} \sum_{j=1}^{\infty} \widetilde{P}_{A,j}\, \zeta_A^{(m)}(Y_j), \quad m = 1, 2, \ldots \tag{41}$$

where $(\widetilde{P}_{A,j})_{j\geq 1}$ is a GEM($H(A)$) sequence, and $(Y_j)_{j\geq 1}$ is an iid sequence of random variables, independent of $(\widetilde{P}_{A,j})_{j\geq 1}$, with common law H_A. To simplify the notation, set $U_{m,j} := \zeta_A^{(m)}(Y_j)$, for any m and j in $\mathbb{N}$. It will now be shown that $(U_{m,j})_{j\geq 1}$ is, for every $m \geq 1$, an infinite exchangeable sequence. To see this, set

$$E_{m,j} := \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^m} \boldsymbol{\epsilon}\, \mathbb{I}_{A_{\boldsymbol{\varepsilon}}}(Y_j), \qquad j \geq 1.$$

For each $m, j \in \mathbb{N}$, $E_{m,j}$ identifies the unique set in the partition Π_m that contains Y_j, for every $j \geq 1$. Since $(Y_j)_{j\geq 1}$ is an iid sequence, then $E_m = (E_{m,j})_{j\geq 1}$ is iid as well, for every m. Now, denote $\widetilde{Z}_\epsilon := Z_{A_\epsilon}$ and, for every m and j in $\mathbb{N}$, set $U_{m,j} = \widetilde{Z}_{E_{m,j}}$. for every choice of positive integers m, n, k and j one, then, has

$$\mathbb{E}\left[\widetilde{Z}^n_{E_{m,1}} \widetilde{Z}^k_{E_{m,j}}\right] = \sum_{\boldsymbol{\varepsilon}_1 \neq \boldsymbol{\varepsilon}_2} \mathbb{E}\left[\widetilde{Z}^n_{\boldsymbol{\varepsilon}_1} \widetilde{Z}^k_{\boldsymbol{\varepsilon}_2}\right] \mathbb{P}\left(E_{m,1} = \boldsymbol{\varepsilon}_1, E_{m,2} = \boldsymbol{\varepsilon}_2\right)$$

$$
\begin{aligned}
&= \sum_{\varepsilon_1 \neq \varepsilon_2} \mathbb{E}\left[\widetilde{Z}^n_{\varepsilon_1}\widetilde{Z}^k_{\varepsilon_2}\right]\mathbb{P}\left(E_{m,1}=\varepsilon_2, E_{m,2}=\varepsilon_1\right)\\
&= \sum_{\varepsilon_1 \neq \varepsilon_2} \mathbb{E}\left[\widetilde{Z}^n_{\varepsilon_2}\widetilde{Z}^k_{\varepsilon_1}\right]\ \mathbb{P}\left(E_{m,1}=\varepsilon_2, E_{m,2}=\varepsilon_1\right)\\
&= \mathbb{E}\left[\widetilde{Z}^k_{E_{m,1}}\widetilde{Z}^n_{E_{m,j}}\right].
\end{aligned}
$$

which, in turn, entails $(\widetilde{Z}_{E_{m,1}}, \widetilde{Z}_{E_{m,j}}) \stackrel{d}{=} (\widetilde{Z}_{E_{m,j}}, \widetilde{Z}_{E_{m,1}})$, for every $j > 1$. This is equivalent to saying that the sequence $(\widetilde{Z}_{E_{m,j}})_{j\geq 1}$ is infinitely exchangeable and so is $U_m = (U_{m,j})_{j\geq 1}$, for any $m \geq 1$. By de Finetti's representation theorem for infinite exchangeable sequences, there is a unique random probability measure τ^0_m on $[0,1]$ conditionally on which the law of U_m is the law of an iid sequence, say $U_m(\tau^0_m)$ with common distribution τ^0_m. Thus, one has

$$
Z_A \stackrel{d}{=} \sum_{j=1}^{\infty} \widetilde{P}_{A,j}\, U_{m,j}(\tau^0_m) \stackrel{d}{=} \int_0^1 x\ \widetilde{p}_{\tau_m}(\mathrm{d}x) \tag{42}
$$

for any $m \geq 1$, where $(\tilde{P}_{A,j})_{j\geq 1}$ and $(U_{m,j}(\tau^0_m))_{j\geq 1}$ are independent and the second distributional equation follows from (39). If B in $\mathscr{X}$ is such that $A\cap B = \varnothing$, a representation for Z_B as in (42) holds true with $\tilde{P}_{A,j}$ replaced by $\tilde{P}_{B,j}$ and $U_{m,j}(\tau^0_m)$ replaced by $U'_{m,j}(\tau'_m)$. Since A and B are disjoint, the two GEM sequences $(\tilde{P}_{A,j})_{j\geq 1}$ and $(\tilde{P}_{B,j})_{j\geq 1}$ are independent. Moreover, given (τ^0_m, τ'_m) the two sequences $\{U_{m,j}(\tau^0_m))\}$ and $\{U'_{m,j}(\tau'_m)\}$ are conditionally independent so that Z_A and Z_B are conditionally independent as well. Furthermore, $\widetilde{p}_{\tau_m}$ is a Dirichlet process on $[0,1]$ with baseline measure $\tau_m := H(A)\tau^0_m$. Hence, using the GEM$(H(A))$ structure of $\widetilde{p}_{\tau_m}$

$$
Z_A \stackrel{d}{=} \widetilde{P}_{m,1}U_{m,1} + (1-\widetilde{P}_{m,1})Z^*_{m,A}, \tag{43}
$$

holds true. In (43), $\widetilde{P}_{m,1}$ is, for every m, a beta$(1, H(A))$ random variable, independent of $(U_{m,1}, Z^*_{m,A})$, and $Z^*_{m,A} \stackrel{d}{=} Z_A$ for every $m = 1, 2, \ldots$. Since (43) is valid for any $m \geq 1$, it can be deduced that $(U_{m,1})_{m\geq 1}$ converges in distribution. With (U_1, Z^*_A) denoting the weak limit of $((U_{m,1}, Z^*_{m,A}))_{m\geq 1}$, one can rewrite (43) as follows

$$
Z_A \stackrel{d}{=} P_{A,1}U_1 + (1-P_{A,1})Z^*_A.
$$

Hence, the array $(U_{m,j})_{j,m\geq 1}$ converges, as $m \to \infty$, to a sequence $U^* = (U^*_n)_{n\geq 1}$ taking value in $[0,1]^\infty$. By Theorem 3.2 in [16], U^* is exchangeable and its directing measure τ^0_A is the weak limit of the directing measures τ^0_m. Summing up,

$$
Z_A \stackrel{d}{=} \sum_j \widetilde{P}_{A,j}U_j = \int_0^1 x\ \widetilde{p}_{\tau_A}(\mathrm{d}x),
$$

where $\tau_A = H(A)\tau_A^0$. Note that the distribution of τ_A does not depend on the choice of nested partitions Π. Hence, we now just have to show the validity of (i)–(ii). To this end, note that from (13)

$$\int_0^1 x\ \widetilde{p}_{\tau_{A\cup B}}(dx) \overset{d}{=} \epsilon_{A,B} \int_0^1 x\ \widetilde{p}_{\tau_A}(dx) + (1-\epsilon_{A,B}) \int_0^1 x\ \widetilde{p}_{\tau_B}(dx)$$

for every pair of disjoint sets A and B in $\mathscr{X}$ and for a random variable $\epsilon_{A,B}$ with beta $(H(A), H(B))$ distribution which is independent from both $\widetilde{p}_{\tau_A}$ and $\widetilde{p}_{\tau_B}$. Moreover, conditionally on (τ_A, τ_B), $\widetilde{p}_{\tau_A}$ and $\widetilde{p}_{\tau_B}$ are also independent of each other. Therefore, conditionally on τ_A, τ_B, the right hand side is equal, in distribution, to $\int_0^1 x\ \widetilde{p}_{\tau_A+\tau_B}(\mathrm{d}x)$. Since the distribution of $\tau_{A\cup B}$ is uniquely defined, then $\tau_{A\cup B} \overset{d}{=} \tau_A + \tau_B$. □

Appendix F: Proof of Corollary 4

The form of the canonical correlations in (22) follows from Theorem 6 the the fact that $Z_i = Z$ for any $i = 1, \ldots, d$. Moreover, given $K = Z\widetilde{\mu}$ one has

$$\mathbb{E}\left[\mathrm{e}^{\int \theta\, \mathrm{d}K}\right] = \mathbb{E}\left[\frac{1}{(1 - Z_\eta \int \theta\, \mathrm{d}\widetilde{p})^c}\right] = \mathbb{E}\left[\frac{1}{(1 - \int \theta\, \mathrm{d}\widetilde{p})^{\eta c}}\right]$$

where the first equality follows from the Markov–Krein identity for random Dirichlet means, and the last one is a well-known hypergeometric identity that can be easily recovered directly by expanding the joint Laplace transform of $(\widetilde{\mu}_1, \widetilde{\mu}_2)$. This shows (23). □

Appendix G: Proof of Corollary 5

In order to identify the canonical correlations we determine an expansion of the Laplace transform of K evaluated at a simple function $f = \sum_{i=1}^d s_i \mathbb{1}_{A_i}$ with sets $A_i \in \mathscr{X}$ being pairwise disjoint. It can then be seen that

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{K(f)}\right] &= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \mathbb{E}\left[\prod_{i=1}^d \frac{K^{n_i}(A_i)}{n_i!} s_i^{n_i}\right] \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^d} \prod_{i=1}^d (cP_0(A_i))_{n_i} \frac{(\eta c P_0(A_i))_{n_i}}{(cP_0(A_i))_{n_i}} \frac{s_i^{n_i}}{n_i!}\end{aligned}$$

and this implies that the canonical correlations are as in (24). Hence the Z_{A_i}'s are independent beta random variables with respective parameters $(\eta c P_0(A_i)$, $(1-\eta)cP_0(A_i))$ and the proof is complete. □

Appendix H: Proof of Theorem 5

Throughout the proof we use the same notation as in the proof of Theorem 4. Let us reconsider the representation $Z_A \stackrel{d}{=} \int_0^1 x\, \widetilde{p}_{\tau_A}(\mathrm{d}x)$, $A \in \mathscr{X}$. If τ_A is, for every A, deterministic, then Z_A and Z_B are independent for every A, B disjoint. Thus K is a CRM and the vector $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ has independent increments by virtue of Theorem 6.

Less trivially, the opposite implication also holds. Assume the vector $(\widetilde{\mu}_1, \widetilde{\mu}_2)$ has independent increments, which is equivalent to K being a CRM, and consider again (41) that leads to the equality in distribution

$$Z_A \stackrel{d}{=} \widetilde{P}_{m,1} U_{m,1} + (1 - \widetilde{P}_{m,1}) Z^*_{m,A},$$

where $U_{m,1} = Z_{A^{(m)}(Y_1)}$. Since cP_0 is diffuse, all the coordinates in the iid sequence $(Y_n)_{n\geq 1}$ are almost surely distinct. Thus, $\lim_{m\to\infty} A^{(m)}(Y_1) \cap A^{(m)}(Y_j) = \varnothing$, almost surely, for every $j \geq 2$. Accordingly, since Z_A and Z_B are independent for every A and B disjoint in $\mathscr{X}$, the almost sure limits $U_1 = \lim_m U_{m,1}$ and $U_j = \lim_m U_{m,j}$ are independent. Hence we obtain that in the limit

$$Z_A \stackrel{d}{=} P_{A,1} U_1 + (1 - P_{A,1}) Z^*_A,$$

where $Z^*_A \stackrel{d}{=} Z_A$ and all the components on the right-hand side are now independent. Due to the independence, a known result of renewal theory (see Lemma 3.3. in [23]) states that such a distributional equation uniquely defines the distribution of Z_A. In this particular case,

$$Z_A \stackrel{d}{=} \sum_j \widetilde{P}_{A,j} U_j$$

where $(U_j)_{j\geq 1}$ is a sequence of iid random variables whose common probability distribution is fixed and denoted as τ^0_A. This is the series representation for a Dirichlet random mean $Z_A = \int_0^1 x\, \widetilde{p}_{\tau_A}(\mathrm{d}x)$, with base measure given by $\tau_A = cP_0(A)\tau^0_A$. Hence, the result follows. □

Appendix I: Proof of Theorem 7.

If $\boldsymbol{X}^{\mathcal{A}}$ has canonical autocorrelations, then Theorem 3 and (26)-(28) imply that, for every t there exists a random vector $\boldsymbol{Z}^{\mathcal{A}}_t = (Z_{A_1,t}, \ldots, Z_{A_d,t}) \in [0,1]^d$ such that

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \mathbb{E}\left[Z^{n_1}_{A_1,t} \cdots Z^{n_d}_{A_d,t} \right].$$

If $H^{\mathcal{A}}_t$ is the law of $\boldsymbol{Z}^{\mathcal{A}}_t$, then

$$\frac{\rho_{\boldsymbol{n}}(\mathcal{A}, t) - \rho_{\boldsymbol{n}}(\mathcal{A}, 0)}{t} = \int_{[0,1]^d} \left(\prod_{i=1}^d z_i^{n_i} - 1 \right) \frac{H^{\mathcal{A}}_t(\mathrm{d}z_1, \ldots, \mathrm{d}z_d)}{t}.$$

In order for the Markov property to hold true, one needs to have $\rho_{\boldsymbol{n}}(\mathcal{A},t) = \exp\{-t\lambda_{\boldsymbol{n}}(\mathcal{A})\}$ for every $t \geq 0$ and $\boldsymbol{n} \in \mathbb{Z}_+^d$. Thus

$$\lambda_{\boldsymbol{n}}(\mathcal{A}) = \lim_{t\downarrow 0} \frac{\rho_{\boldsymbol{n}}(\mathcal{A},0) - \rho_{\boldsymbol{n}}(\mathcal{A},t)}{t}$$

For every t consider the measure $G^{\mathcal{A}}(\cdot,t)$ defined by

$$G^{\mathcal{A}}(B,t) := \int_B \left(1 - \prod_{i=1}^d z_i\right) \frac{H_t^{\mathcal{A}}(\mathrm{d}z_1,\ldots,\mathrm{d}z_d)}{t},$$

for every Borel set B of $[0,1]^d$. Note that, since $\rho_{\boldsymbol{n}}(\mathcal{A},0) = 0$ for every $\boldsymbol{n}$, $G^{\mathcal{A}}\left([0,1]^d,t\right) = t^{-1}\left(1-e^{-t\lambda_{\mathbf{1}}(\mathcal{A})}\right)$, where $\mathbf{1} = (1,1,\ldots,1)$. Thus $G(\cdot,t)$ has, for every $t \leq \delta$, say, a total mass that is uniformly bounded by a constant that only depends on δ. By the continuity of the function

$$\frac{1-\prod_{i=1}^d z_i^{n_i}}{1-\prod_{i=1}^d z_i},$$

it is then routine to show that $G^{\mathcal{A}}(\cdot,t)$ converges, as $t \downarrow 0$, to a finite measure $G^{\mathcal{A}}$ on $[0,1]^\infty$, hence, denoted $H^{\mathcal{A}}(\mathrm{d}\boldsymbol{z}) = \left(1-\prod_{i=1}^d z_i\right) G^{\mathcal{A}}(\mathrm{d}z_1\cdots\mathrm{d}z_i)$, we see that

$$\lambda_{\boldsymbol{n}}(\mathcal{A}) = \int_{[0,1]^d} \left(1-\prod_{i=1}^d z_i^{n_i}\right) H^{\mathcal{A}}(\mathrm{d}\boldsymbol{z})$$

The change of variable $y_i = -\log z_i$, along with the corresponding change of measure $\widetilde{G}^{\mathcal{A}}(B) = G^{\mathcal{A}}\left(\{\boldsymbol{z} \in [0,1]^d : (-\log z_1,\ldots,-\log z_d) \in B\}\right)$ for any B in $\mathscr{B}(\mathbb{R}_+^d)$, leads to

$$\rho_{\boldsymbol{n}}(\mathcal{A},t) = \exp\left\{-t\int_{\mathbb{R}_+^d} \left(1-\mathrm{e}^{\sum_{i=1}^d n_i y_i}\right) \widetilde{G}_A(\mathrm{d}\boldsymbol{y})\right\}. \tag{44}$$

This, in turn, implies that

$$\boldsymbol{Y}^{\mathcal{A}} = \{(-\log Z_{A_1,t},\ldots,-\log Z_{A_d,t}) : t \geq 0\}$$

is a d-dimensional subordinator with Lévy measure $\widetilde{G}_A$.

To prove the converse, construct a Markovian sequence $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$ of d–dimensional random vectors such that for every $j = 0,1,\ldots$, $(\boldsymbol{X}_j, \boldsymbol{X}_{j+1})$ is a bivariate gamma$(cP_0(A_1),\ldots,cP_0(A_d))$ random pair of vectors with canonical correlations $\rho_{\boldsymbol{n}}(\mathcal{A}) = \int_{[0,1]^d} \prod_{i=1}^d z_i^{n_i} F^{\mathcal{A}}(\mathrm{d}\boldsymbol{z})$ for some measure F_A. It is easy to see that, from the symmetry of the one-step kernel, the chain is reversible and that

$$\mathbb{E}\left[\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{X}_k) \mid \boldsymbol{X}_0 = \boldsymbol{x}\right] = \rho_{\boldsymbol{n}}^k(\mathcal{A})\widetilde{L}_{\boldsymbol{n},\boldsymbol{\alpha}}(\boldsymbol{x}), \qquad k = 0,1,\ldots$$

that is $(\boldsymbol{X}_0, \boldsymbol{X}_k)$ have canonical correlations $\rho_{\boldsymbol{n}}(\mathcal{A}, k) = \rho^k_{\boldsymbol{n}}(\mathcal{A})$ for any k. Consider now a Poisson process $\{N_t : t \geq 0)\}$ on $\mathbb{R}_+$ with rate γ, independent of $(\boldsymbol{X}_k)_{k\geq 0}$ and set

$$\tilde{\boldsymbol{X}}(t) := \boldsymbol{X}_{N(t)}, \qquad t \geq 0.$$

By construction, $\{\tilde{\boldsymbol{X}}_t : t \geq 0\}$ is again a Markov process and has canonical autocorrelations

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \mathbb{E}\left[\rho^{N(t)}_{\boldsymbol{n}}(\mathcal{A})\right] = \exp\left\{-t\gamma\left[1 - \rho_{\boldsymbol{n}}(\mathcal{A})\right]\right\}$$

But this can also be written as

$$\gamma\left[1 - \rho_{\boldsymbol{n}}(\mathcal{A})\right] = \int_{[0,1]^d} \left(1 - \prod_{i=1}^d z_i^{n_i}\right) G^{\mathcal{A}}(\mathrm{d}\boldsymbol{z})$$

where $G^{\mathcal{A}} := \gamma F^{\mathcal{A}}$. This property is preserved even as $\gamma \to \infty$, if the limit measure, with respect to the topology of the weak convergence, $G^{\mathcal{A}} := \lim_{\gamma\to\infty} \gamma F^{\mathcal{A}}$ exists. So, after the change of variable $y_i = -\log z_i$, denoting with $\tilde{G}^{*\mathcal{A}}$ is the corresponding induced measure on $\mathbb{R}^d_+$,

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \exp\left\{-t \int_{[\mathbb{R}^d_+} \left(1 - e^{-\sum_1^d n_i y_i}\right) \tilde{G}^{*\mathcal{A}}(\mathrm{d}\boldsymbol{y})\right\}$$

is, for every t, a canonical correlation sequence and, since it satisfies (26)-(28), it is a Markov canonical correlation sequence. □

Appendix J: Proof of Theorem 8.

Suppose $\xi = \{\xi_t : t \geq 0\}$ is a time-homogeneous, reversible, measure–valued Feller process with Γ_{cP_0} stationary measure and canonical autocorrelations ρ. Then by Proposition 2, for every t, every d and every $\mathcal{A} = (A_1, \ldots, A_d) \in \mathscr{X}^*$, the canonical correlations of $(\xi_0(A_1), \ldots, \xi_0(A_d))$ and $(\xi_t(A_1), \ldots, \xi_t(A_d))$, must satisfy

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \mathrm{e}^{-t\psi_{\mathcal{A}}(\boldsymbol{n})} = \mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^d n_i Y_{A_i,t}}\right], \qquad \boldsymbol{n} \in \mathbb{Z}^d_+ \tag{45}$$

for a multivariate subordinator $\boldsymbol{Y}^{\mathcal{A}}$ with Laplace exponent $\psi_{\mathcal{A}}$. Now, from Theorem 3(iii), for every $A \in \mathscr{X}$ and every measurable partition A_0, A_1 of A,

$$\begin{aligned}
\rho_n(A, t) &= \rho_n(A_0 \cup A_1, t) \\
&= \sum_{j=0}^n \binom{n}{j} \frac{(cP_0(A_0))_j (cP_0(A_1))_{n-j}}{(cP_0(A))_n} \rho_{j,n-j}(A_0, A_1, t) \\
&= \mathbb{E}\left[\rho_{J,n-J}(A_0, A_1, t)\right]
\end{aligned} \tag{46}$$

where J is a random variable whose distribution is beta–binomial distribution with parameter $(n; cP_0(A_0), cP_0(A_1))$. But by virtue of (29), $\rho_n(\mathcal{A}, t) = \rho_n^t(\mathcal{A}, 1)$ for any $t \geq 0$, which entails

$$\rho_n^t(A, 1) = \mathbb{E}\left[\rho_{J,n-J}^t(A_0, A_1, 1)\right] \qquad \forall t \geq 0.$$

This equality for $t = 2$ shows that that the random variable

$$\rho_{J,n-J}(A_0, A_1, 1) = \mathbb{E}\left[\mathrm{e}^{-JY_{A_0,1}-(n-J)Y_{A_1,1}} \mid J\right]$$

has variance 0, namely it is almost surely constant. Hence $Y_{A_0,1} = Y_{A_1,1}$ almost surely. By Theorem 3 this must hold true for every choice of A and of a partition $\{A_0, A_1\}$ of A. This argument also leads to state that $Y_{A,1} = Y_{A^c,1} = Y_{\mathbb{X},1}$ almost surely for every A. Thus, for every A, $\rho_n(A, 1) = \mathbb{E}\left[e^{-nY_{\mathbb{X},1}}\right] = \rho_n(\mathbb{X}, 1)$ and

$$\rho_{j,n-j}(A, A^c, 1) = \mathbb{E}\left[e^{nY_{\mathbb{X},1}}\right] = \rho_n(\mathbb{X}, 1).$$

Since $\rho_n(\mathbb{X}, t) = \rho_n^t(\mathbb{X}, 1)$ for every $t \geq 0$, this, combined with (29), is enough to conclude that there exists a one–dimensional subordinator $S = \{S_t : t \geq 0\}$ such that

$$\rho_{\boldsymbol{n}}(\mathcal{A}, t) = \mathbb{E}\left[e^{-|\boldsymbol{n}|S_t}\right], \qquad \boldsymbol{n} \in \mathbb{Z}_+^d,$$

for every collection $\mathcal{A} = \{A_1, \ldots, A_d\}$ of pairwise disjoint sets in $\mathscr{X}$. This shows just what we wanted to prove: the process ξ with canonical correlations in this form has the same law as a subordinated Dawson-Watanabe process. The last part of the Theorem (the generator of ξ) is a direct application of [22], Theorem 32.1. □

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