Evaluating stationarity via epidemic change-point alternatives for fMRI data

Claudia Kirch

Karlsruhe Institute of Technology (KIT)

joint work with John Aston (University of Warwick)

Recent Advances in Changepoint Analysis, 26-28 March, 2012

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Measures blood flow in the brain associated with neural activity

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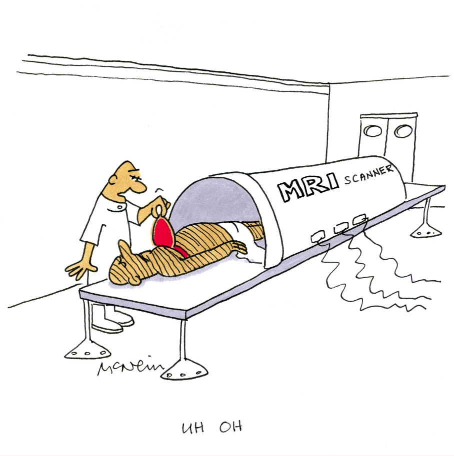

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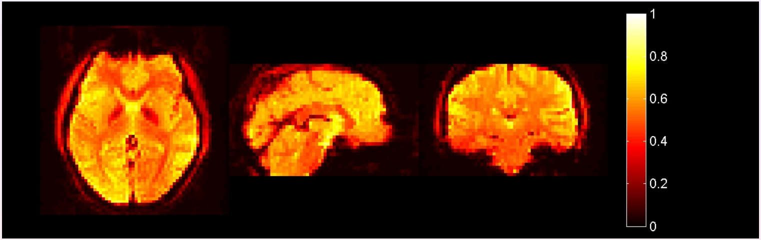

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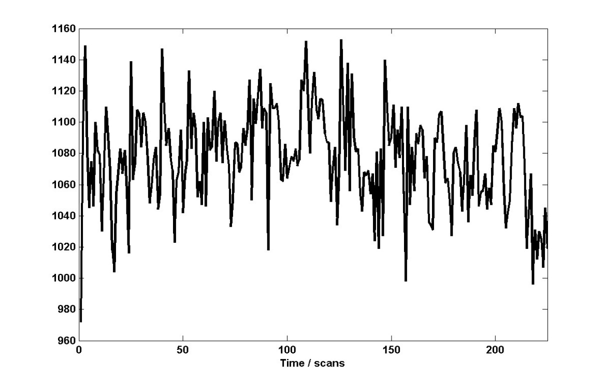

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Dimension reduction via orthonormal system $\{v_k(\cdot), k = 1, \ldots, d\}$:

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Each scan:

225 time points of a 3-dim. image of $64 \times 64 \times 33 \approx 10^5$ voxels.

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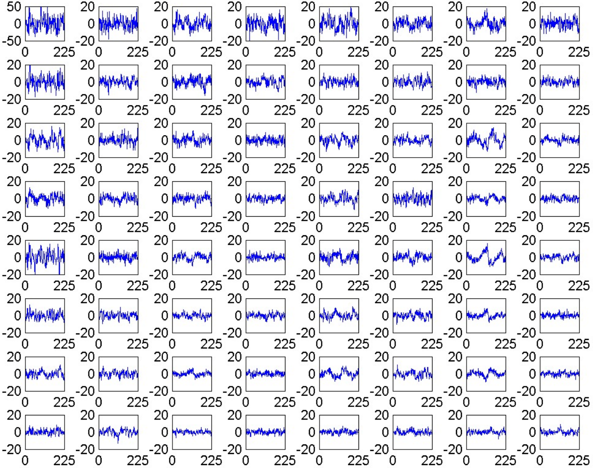

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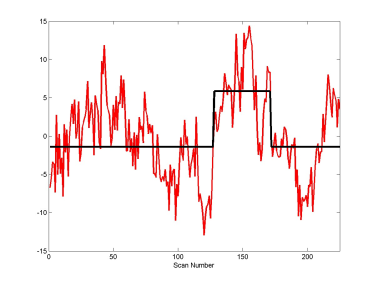
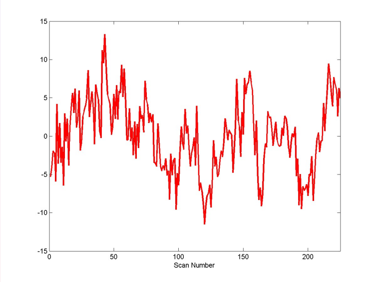


Epidemic change removed:



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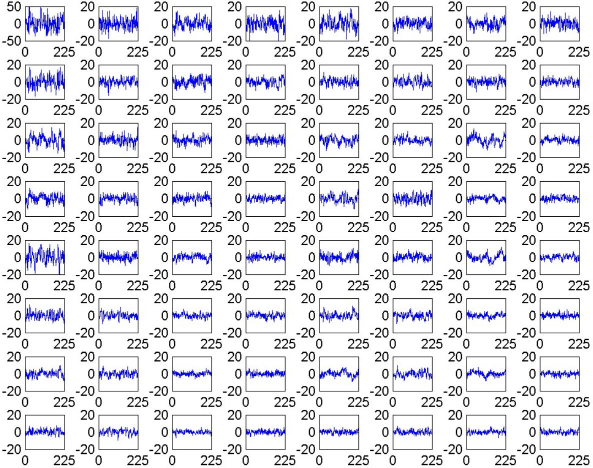

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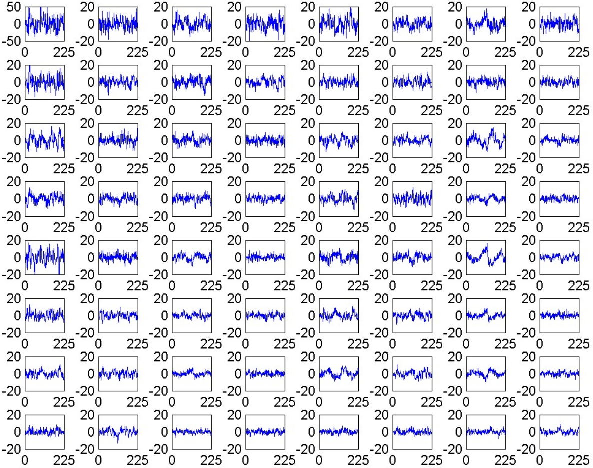
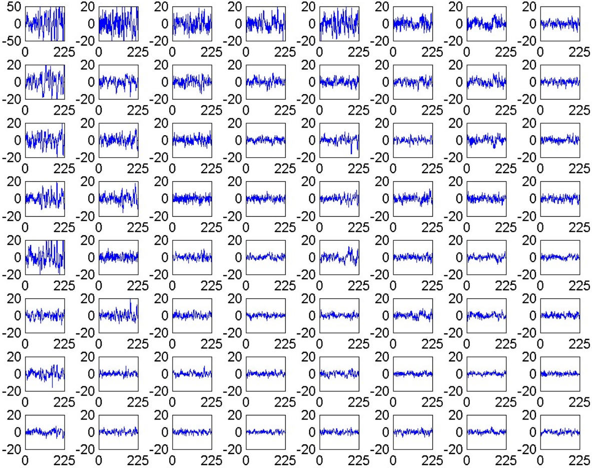


Resting State Example 2: Medium level shift



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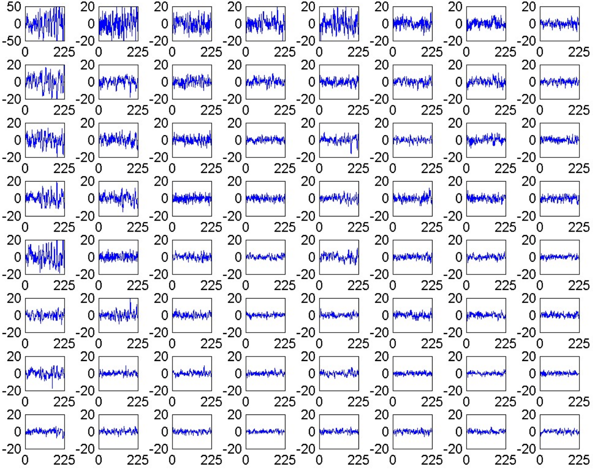
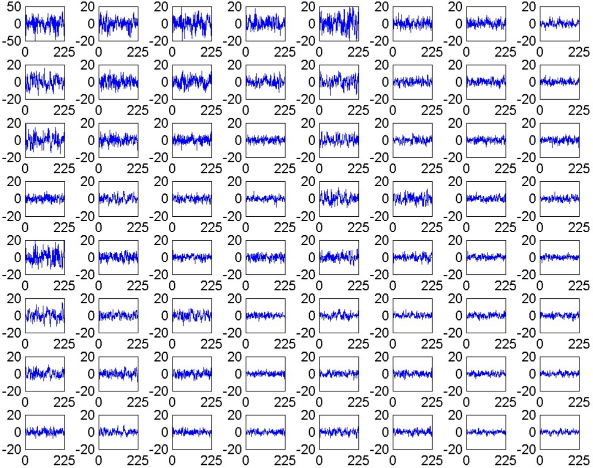


Resting State Example 3: No level shift?



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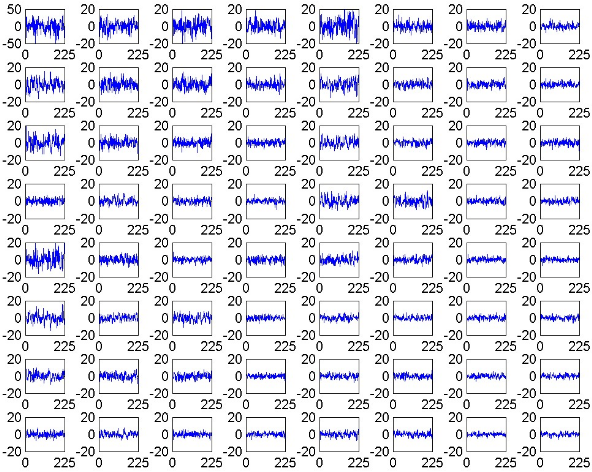

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Test statistic for epidemic change:

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$d \times d$ matrix ($d = 64, 125$) but only 225 time points:

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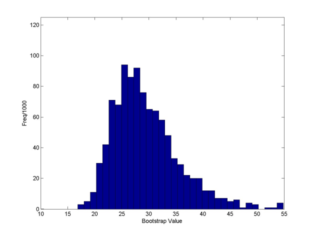

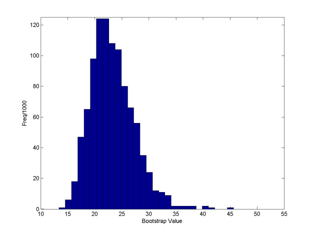

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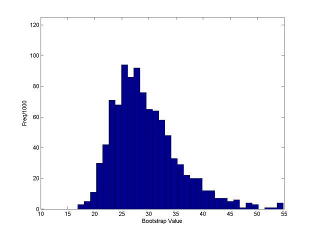
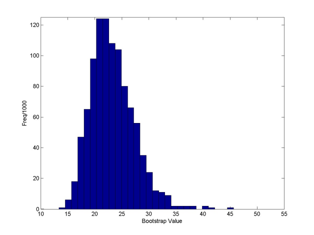
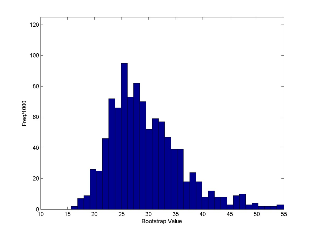
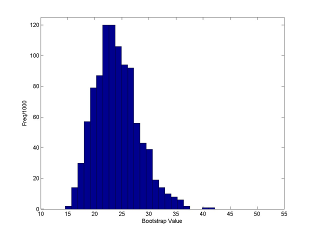




Bootstrap distributions
with change detected





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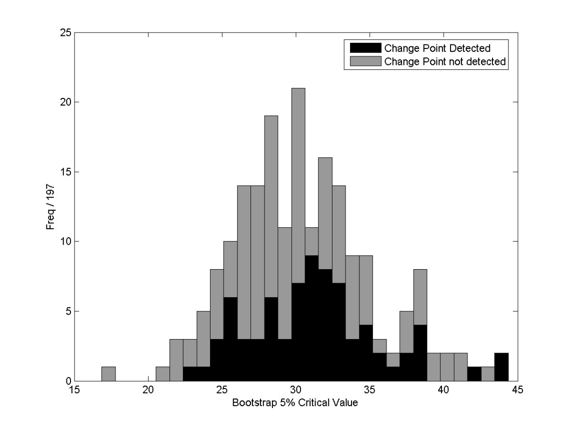

Distribution of bootstrap 5% critical values from 197 scans

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Very high-dimensional data set (10^5 voxel),
very short time series (225 time points):

$\Longrightarrow$ inaccurate estimate of principal components (e.g. $\leqslant 225$ non-zero eigenvalues).

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Parametric assumption: Separable covariance structure:

$c((t_1, t_2, t_3), (s_1, s_2, s_3)) = c_1(t_1, s_1)\, c_2(t_2, s_2)\, c_3(t_3, s_3).$

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Contaminated subspace:

$\{w_k\}$ eigenfunctions of $c(t,s) + \theta(1-\theta)\Delta(t)\Delta(s)$, $\theta = \vartheta_2 - \vartheta_1$

($d+1$ distinct eigenvalues needed).

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Test results:

- $64\,(=4^3)$ components: 78 (of 197) reject at 5% level
  70 after FDR-Correction, FDR threshold: 0.022
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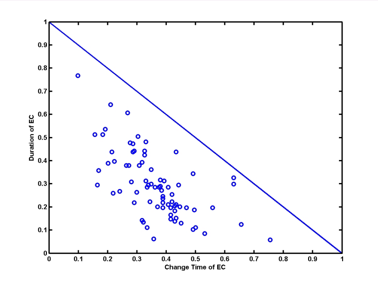
Observations of several independent time series (persons) involving epidemic changes.
Of interest: Delay and duration distribution!



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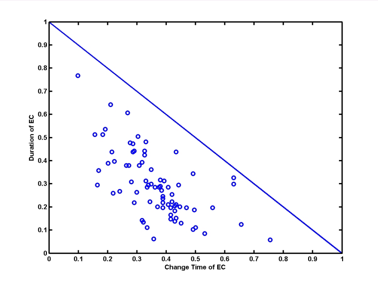

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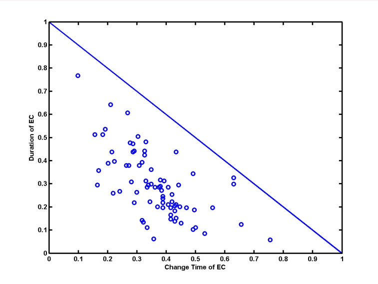

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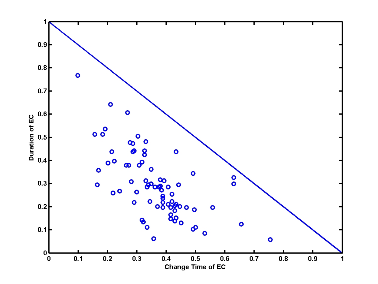

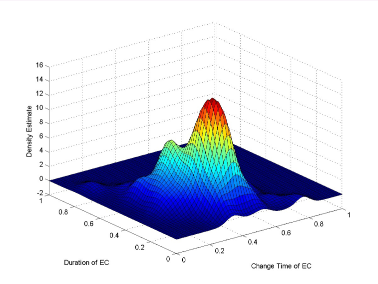

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