Stochastic Volatility Model with Leverage

Stochastic Volatility (SV) models are an increasingly popular choice for modelling financial time-series data. The basic SV model assumes an autoregressive structure for the log-volatility, and it is able to match the empirically observable low serial autocorrelation in the squared return series. The SV model with leverage (SVL) extends the SV model by allowing the return series and the increment of the log- volatility series to correlate. This correlation models a real world phenomenon, the asymmetric return volatility.

Formally, in the centered parameterization, for $t = 1, \ldots, T$, \( y_t = \exp(h_t z_t) \) with \( (h_t, z_t) \sim N(\mu, \rho, \sigma) \), a Gaussian mixture distribution can be obtained by substituting \( h_t = (h_t - \mu) / \sigma \) into (C).

\[ y_t = \exp(h_t z_t) \]

**Auxiliary sampler**

In terms of sampling vol. efficiently, the state-of-the-art MCMC sampler is based on a auxiliary model developed in [3]. It transforms the observation equation to a linear form, and then approximates \( (h_t, z_t) \) by a ten component mixture of bivariate Gaussian distributions. With the mixture components denoted by \( s_i \in \{1, 10\} \), the auxiliary model in NC is

\[ \log(y_t) = \mu h_t^T m_i + v_h^T w_h. \]

and also its derivatives can be evaluated. An independent Metropolis-Hastings (MH) step is used for \( (\mu, \rho, \sigma) \), and then Gaussian simulation smoothing for \( h_t \) and the vector \( h_t \).

**Direct sampler**

Direct estimation of (C) or (NC) is also possible since \( (\mu, \rho, \sigma) \sim N(\mathbf{x}, \mathbf{I}) \), and also its derivatives can be evaluated. Due to the issues with the independent MH sampler for \( h_t \), we tried the random-walk MH (RWWM) approach, and the Metropolis adjusted Langevin algorithm (MALA) for parameter sampling, and, as an approximation, stayed with the efficient simulation smoother from the (A) sampler. This approximation can be corrected by an MH acceptance-rejection step.

As already shown for the basic SV model [2], samplers based on different parameterizations can have substantially different sampling efficiency on the same data set due to the altered dependence structure. To exploit this phenomenon, the ancillarity-sufficiency interweaving strategy (ASIS) [5] can utilize samples of both C and NC, and thus ASIS may be able to deliver a markedly higher effective sample size than C or NC samplers. ASIS affects only dependent MH algorithms, hence we took advantage of it in the RWWM and the MALA samplers.

**Stan & JAGS**

For completeness, Stan [1] and JAGS [4] were also tried out through their R interface in both the C and the NC parameterizations.

**Setup**

The samplers below were run on an extensive grid of parameters, altogether 91500 different MCMC chains were run. The length of the burn-in was 5000, and 50000 samples were drawn afterwards. The initial values were the true ones in all cases, and the priors were always

\[ \mu = 1, \psi = 2/3, \beta = 2 \]

\[ \psi = 2, \beta = 3, \gamma = 2/3, \tau = 0.5 \]

\[ \sigma = 10^{0.01 \tilde{h}} \quad \tilde{h} = N(0, \psi^2 (1/\tau^2)) \]

**Efficiency of \( \phi \) and \( \psi \)**

In general, the picture looks similar to the case of \( \phi \) and \( \psi \). Stan outperforms all other choices in terms of ESS. If runtime is also of concern, then RWWM is the strongest choice for small data sets, while MALA together with RWWM show the best performance for larger data sets.

**Efficiency of the volatility**

Interestingly, the general framework of Stan is able to deliver the highest ESSs, slightly outperforming on average even the model-specific, optimized auxiliary sampler. RWWM with ASISs 4 to 8 times smaller ESS for the latent vector than Stan in most cases. In terms of ESS, Stan and JAGS are the least favorable, while RWWM and MALA without ASISs perform ca. 10 times better than other choices. Since the Gaussian simulation smoother is a highly efficient algorithm both in terms of speed and sampling efficiency, the computation times of the \( (\phi, \psi, \omega) \) draws becomes a crucial factor, in which RWWM excels the most.

**Effect of reparameterization and ASIS**

According to the Table, the sampling efficiency of JAGS, Stan, and AUX greatly depends on the parameterization and the data generating process. This effect is observable only in a weaker form at RWWM and MALA. On the other hand, ASIS significantly increases both the ESS and the ASR, and we recommend ASIS for more reliable performance as well.

**Future**

The most promising algorithm would be included in the R package `rstan`, as a computationally highly efficient, compiled extension to the basic SV model sampler.