Quantitative convergence properties of simple slice sampling
Viacheslav Natarovskii & Daniel Rudolf

Problem

For $G \subseteq \mathbb{R}^d$ and $\varphi: G \to [0, \infty)$ we want to sample with respect to

$$p(A) = \int_G p(x) dx$$

by the simple slice sampler.

Qualitative convergence results, see e.g. [1].

Simple slice sampler

The simple slice sampler generates a Markov chain $(X_n)_{n \in \mathbb{N}}$ as follows:

1. Draw $T_n \sim \text{Unif}(0, p(x_n)]$, call the result $t_n$;
2. Draw $X_{n+1} \sim \text{Unif}(G(t_n))$, call the result $x_{n+1}$.

Notations and basics

- The level set of $p$ (of level $t$):
  $$G(t) := \{ x \in G \mid p(x) \geq t \}, \quad t \in [0, \| p \|_{L^\infty}]$$

- The transition kernel of $(X_n)_{n \in \mathbb{N}}$:
  $$U(x, A) = \frac{1}{p(x)} \int_A \frac{p(y) \text{vol}_d(A \cap G(y))}{\text{vol}_d(G(y))} dy, \quad A \subseteq G$$

- The transition kernel of $(T_n)_{n \in \mathbb{N}}$:
  $$Q(t, B) = \frac{1}{\text{vol}_d(G(t))} \int_{G(t)} \frac{\text{vol}_d(B \cap \{ 0, p(x) \})}{p(x)} dx, \quad B \subseteq (0, \infty)$$

- The probability measure $\mu$ on $(0, \infty)$:
  $$\mu(B) = \int_B \frac{\text{vol}_d(G(x))}{\int_G \text{vol}_d(G(t)) dt} dt, \quad B \subseteq (0, \infty)$$

- The Markov operators $U$ and $Q$:
  $$U(\alpha)(x) := \int_G f(y) U(x, dy), \quad Q(\alpha) := \int_0^\infty g(\alpha) Q(t, ds)$$

Spectral gap

The spectral gaps of $U$ and $Q$ are defined by

$$\text{gap}(U) := 1 - \| U - E_\varphi \|_{L^2(\mu) \to L^2(\mu)}, \quad \text{gap}(Q) := 1 - \| Q - E_\varphi \|_{L^2(\mu) \to L^2(\mu)}$$

If $\text{gap}(U) > 0$, then

- $\| U\varphi(y) - \pi(y) \|_\infty \leq (1 - \text{gap}(U))^n \| \varphi \|_{L^2} - 1 \|_2$;
- a CLT holds for the ergodic average for all $f \in L^2(\pi)$, see [2];
- there is a mean squared error bound for the ergodic average for $f \in L^p(\pi), p > 2$, see [3].

Main tool: Wasserstein metric

- Wasserstein distance between two measures on $G$:
  $$W(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \int_G |x - y| d\gamma(x, y),$$
  where $\Gamma(\mu, \nu)$ is the set of all couplings of $\mu$ and $\nu$;
- By [4, Proposition 30] we have
  $$W(U(x, \cdot), U(\varphi, \cdot)) \leq \alpha |x - \varphi|, \quad \forall x, \varphi \in G \implies \text{gap}(U) \geq 1 - \alpha.$$

Main results

Theorem 1 (Natarovskii, Rudolf, 2018).

Let $\varphi(x) = e^{-\varphi(x)}$, where $\varphi : [0, \infty) \to [0, \infty)$ satisfies the following conditions:

1. $\varphi \in C^1([0, \infty))$,
2. $\varphi(0) \geq 0$ and $\varphi'(s) > 0$ for all $s > 0$,
3. $\varphi(s)$ is non-decreasing.

Then

$$W(U(x, \cdot), U(\varphi, \cdot)) \leq \frac{d}{d + 1} |x - \varphi|, \quad \forall x, \varphi \in G$$

Lemma 2 (Natarovskii, Rudolf, 2018).

Let $\varphi$ be such that for some $k \in \mathbb{N}$ the function $\varphi(s) := -\log \left( \int \left\{ t : \text{vol}_d(G(t)) \leq \text{vol}_d(B_k^2 t^2) \right\} \right)$ satisfies conditions 1.-3. from Theorem 2. Then

$$\text{gap}(U) \geq 1 - \frac{k}{k + 1} = \frac{1}{k + 1}.$$

Illustration of the technique

Let $\varphi(x)$ be the maximum of two Gaussian densities:

Examples of densities with the same level-set function as $\varphi(x)$:

Future research

- Weakening assumptions of Theorem 1;
- Spectral gap for elliptical slice sampling, see [5];
- Spectral gap for hybrid slice sampling, see [6].

References

2. Kipnis, Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Communications in Mathematical Physics 104 (1986), no. 1, 1–19.