The ZigZag Process

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In the last few years there is a tendency to use non-reversible Markov Processes to simulate from the invariant distribution.

A new class of processes used is Piecewise Deterministic Markov Processes.

They follow deterministic dynamics for some random period of time. Then they switch to different dynamics.

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The Zig-Zag process in $\mathbb{R}$

In 1-dimension, the process moves with constant speed $+1$ to the right and then changes to the left etc.

The state space is $\mathbb{R} \times \{-1, +1\}$

The randomness is formulated in terms of a Poisson Process which is characterized by a function $\lambda(x, \theta)$.

If we start from $(0, +1)$, continue in this direction until time $T_1$. $T_1$ is the first event of a Poisson Process with intensity $\{\lambda(t, +1), t \geq 0\}$.

If we change at $T_1$ continue towards the left and change again at time $T_1 + T_2$ where $T_2$ is the first point of a Poisson Process with intensity $\{\lambda(T_1 - t), \tau \geq 0\}$.
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The Zig-Zag process in \( \mathbb{R} \)

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If we start from \((0, +1)\), continue in this direction until time \( T_1 \). \( T_1 \) is the first event of a Poisson Process with intensity \( \{\lambda(t, +1), t \geq 0\} \).

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**Figure:** Normal Distribution  
Picture by J. Bierkens
Figure: Cauchy Distribution
Picture by J. Bierkens
Invariant Measure

All randomness is hidden in $\lambda$.

**Proposition**

If $\pi(dx, d\theta) = \frac{1}{Z} \exp\{-U(x)\}(dx, d\theta)$

then the Zig-Zag process with rates

$$\lambda(x, \theta) = (\theta U'(x))^+ + \gamma(x), \ \gamma \geq 0$$

has $\pi$ as unique invariant measure.
Theorem (Bierkens-Roberts-Zitt (2017))

Consider \((X_t, \Theta_t)\) in \(\mathbb{R}^d\). If \(U \in C^3\), there exist \(c > d, c'\) with \(U(x) \geq c \log(1 + |x|) - c'\) and \(U\) has a non-degenerate local minimum, then

\[
\lim_{t \to \infty} \sup_{A} \| \mathbb{P}_{X, \Theta}[(X_t, \Theta) \in A] - \pi(A) \|_{TV} = 0.
\]

This implies that for all \(f \in L^1(\pi)\),

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\frac{1}{T} \int_0^T f(X_t)dt \to E_{\pi}[f(X)] \text{ as } T \to \infty \text{ a.s.}
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Central Limit Theorem

Theorem (Bierkens-Roberts-Zitt (2017))

Under some heavier conditions on the growth of $U$ and assuming that it has lighter tails, we also have a CLT for the Zig-Zag

$$\frac{1}{\sqrt{n}} \int_0^n f(X_s, \Theta_s) - \pi(f)ds \to \mathcal{N}(0, \sigma_f^2)$$

in distribution as $n \to \infty$, for some $0 \leq \sigma_f < \infty$. 
My research

Why restrict to \(\{-1, +1\}^d\) velocities?

Allow \(\{-\theta_m, ..., -\theta_1, \theta_1, .., \theta_m\}^d\).

Need more Poisson Processes, for any pair of possible speeds \(\theta_i, \theta_j\) need a function \(\lambda(x, \theta_i, \theta_j)\).

More freedom to choose the algorithm. However seems to be heavier computationally.

When we only allow jumps to "neighbouring" speeds, the Ergodicity Theorem seems to hold, using basically the same arguments.