Exact Simulation of the Supremum of a Stable Process

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The Problem

- Simulation of the supremum of Lévy processes is a tough problem:
  - Few cases with exact simulation in the infinite activity case (even worse for the infinite variation case!)
  - It is not known how good discretisations are.
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  - Few cases with exact simulation in the infinite activity case (even worse for the infinite variation case!)
  - It is not known how good discretisations are.
- **Stable** processes:
  - Often used as classical examples because their self-similarity often allow for **closed form** formulas.
  - Even here, only spectrally one-sided cases seem feasible from the literature [BDP11]
The Ingredients and the Strategy

Supremum of Stable Processes (slide 3)
The Main Idea

$D_0$

$\sup_{n}$
The Main Idea

\[ D_{-1} \quad n \]

\[ D_0 \]
The Main Idea

\[ D_0, D_{-1}, D_{-2} \]

\[ n \]
The Main Idea
The Main Idea
The Main Idea
The Ingredients and the Strategy

- Exponential Bounds
- Dominating Process $D_n \geq X_n$
- RW & Reflected Process
- MC $\{X_n\}$ with stationary law $\overline{Y}_1$
- Perpetuity Equation
- Update Function
- Detect $-\sigma$
- Sample $X_0$

Supremum of Stable Processes (slide 10)
Outline of the Talk

Stochastic Perpetuity

Markov Chain

Dominating Process
Stable Processes

Using Zolotarev’s (C) form, given any $\alpha \in (0, 2]$ and any skewness parameter $\beta \in [-1, 1]$

$$\rho = \mathbb{P}(Y_1 > 0) = \frac{\theta + 1}{2}, \quad \theta = \beta \left(\frac{\alpha - 2}{\alpha} 1_{\alpha > 1} + 1_{\alpha \leq 1}\right),$$

then

$$\log \left(\mathbb{E} \left( e^{itY_1} \right) \right) = -|t|^\alpha e^{-i \frac{\pi \alpha}{2} \theta \text{sgn}(t)}.$$
Concave Majorant

Fix a Lévy process \( \{ Y_t \} \). Its concave majorant is the (random) smallest concave function \( \{ C_t \} \) that dominates \( \{ Y_t \} \).
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Concave Majorant

Discover the faces of $C$ independently at random, uniformly on lengths. Then the faces satisfy [PUB12]:

\[
\{(d_n - g_n, C_{d_n} - C_{g_n})\}_{n} \overset{d}{=} \{(l_n, Y_{L_n} - Y_{L_{n-1}})\}_{n} \overset{d}{=} \left\{ \left( l_n, \frac{1}{l_n} Z_n \right) \right\}_{n},
\]

for independent iid $U_n \sim U(0,1)$, $l_n = U_n (1 - L_{n-1})$, and $L_n = \sum_{i=1}^{n-1} l_i$ (stick-breaking process) and an independent iid sequence $Z_n \overset{d}{=} Y_1$. Then $\bar{Y}_1 := \sup_{t \in [0,1]} Y_t$ satisfies

\[
\bar{Y}_1 = \sum_{n=1}^{\infty} \ell_n^\frac{1}{\alpha} Z_n^+ = \ell_1^\frac{1}{\alpha} Z_1^+ + (1 - \ell_1)^\frac{1}{\alpha} \sum_{n=2}^{\infty} \left( \frac{l_n}{1 - \ell_1} \right)^\frac{1}{\alpha} Z_n^+ 
\]

\[
\overset{d}{=} U^\frac{1}{\alpha} \bar{Y}_1 + (1 - U)^\frac{1}{\alpha} Z_1^+.
\]
Stochastic Perpetuity

Let $S^+(\alpha, \rho)$ and $\overline{S}(\alpha, \rho)$ be the laws of $Y_1$ conditioned on being positive and of $\overline{Y}_1$ respectively. Then, the relation for the faces of $C$ and the scaling property of stable processes then yield [GCMUB18]:

$$\overline{Y}_1 \overset{d}{=} \left(1 + B \left(V^{\frac{1}{\alpha \rho}} - 1\right)\right) \left(U^{\frac{1}{\alpha}} \overline{Y}_1 + (1 - U)^{\frac{1}{\alpha}} S\right),$$

where $(B, U, V, S, \zeta) \sim \text{Ber}(\rho) \times U(0, 1)^2 \times S^+(\alpha, \rho) \times \overline{S}(\alpha, \rho)$. And $\overline{S}(\alpha, \rho)$ is the unique solution.
First Update Function

Let $\Theta = (U, W, \Lambda, S)$ for an independent $W \sim U(0,1)$. Then the perpetuity may be summarised as

$$\overline{Y}_1 \overset{d}{=} \phi(\overline{Y}_1, \Theta),$$

where $\Lambda = 1 + B \left( V^{1/\rho} - 1 \right)$ and

$$\phi(x, \theta) = \lambda^{\frac{1}{\alpha}} \left( u^{\frac{1}{\alpha}} x + (1 - u)^{\frac{1}{\alpha}} s \right).$$

Consider the functions

$$a(\theta) = \left( \lambda^{-\frac{1}{\alpha}} - 1 \right) u^{-\frac{1}{\alpha}} (1 - u)^{\frac{1}{\alpha}} s,$$

$$\psi(x, \theta) = 1_{\{x \leq a(\theta)\}} W^{\frac{1}{\alpha \rho}} (1 - u)^{\frac{1}{\alpha}} s + 1_{\{x > a(\theta)\}} \phi(x, \theta).$$
Update Function

Update Functions

\[ \phi(x, \theta) \]

\[ (1 - u)^{\frac{1}{\alpha}} s \]

Supremum of Stable Processes (slide 18)
Update Functions

\[ \phi(x, \theta) \]

\[ \psi(x, \theta) \]

\[ w^{\frac{1}{\alpha \rho}} (1 - u) \left( \frac{1}{\alpha} s \right) \]

\[ a(\theta) \]

Supremum of Stable Processes (slide 19)
Second Update Function

- Then $X \sim \overline{S}(\alpha, \rho)$ is the unique solution to
  \[ X \overset{d}{=} \psi(X, \Theta). \]

- The difference between $\phi$ and $\psi$ is that the latter has positive probability of ignoring the specific value of $X$. 
Consider a Markov chain on stationarity \( \{X_n\}_{n \in \mathbb{Z}} \) driven by the i.i.d. sequence \( \{\Theta_n\}_{n \in \mathbb{Z}} \) satisfying
\[
X_{n+1} \overset{d}{=} \psi (X_n, \Theta_n).
\]
If we were able to find a time \( -\tau < 0 \) such that \( X_{-\tau} \leq a (\Theta_{-\tau}) \), then
\[
X_0 = \psi \left( \cdots \psi \left( W_{-\tau}^{\alpha \rho} (1 - U_{-\tau})^{\frac{1}{\alpha}} S_{-\tau}, \Theta_{-\tau+1} \right), \cdots , \Theta_{-1} \right),
\]
\( \tau - 1 \) times
so we can compute \( X_0 \sim \mathcal{S} (\alpha, \rho) \) from \( \{\Theta_n\}_{n \in \{-\tau, \ldots, -1\}} \).
Recall that if $\tau_n$ is the last time $\{X_{n-k} \leq a(\Theta_{n-k})\}$, then

$$X_n = \sum_{k=\tau_n+1}^{n-1} e^{\frac{1}{\alpha} \sum_{j=k+1}^{n-1} \log(\Lambda_j U_j)} \Lambda_k^{\frac{1}{\alpha}} (1 - U_k)^{\frac{1}{\alpha}} S_k$$

$$+ e^{\frac{1}{\alpha} \sum_{j=\tau_n+1}^{n-1} \log(\Lambda_j U_j)} W_\tau^{\frac{1}{\alpha \rho}} (1 - U_\tau)^{\frac{1}{\alpha}} S_\tau$$

$$\leq e^{R_n} \sum_{k=-\infty}^{n-1} e^{-(n-1-k)\delta} (1 - U_k)^{\frac{1}{\alpha}} S_k$$

$$\leq e^{R_n} \left( \frac{e^{(d-\delta)\chi_n-n}}{1 - e^{\delta-d}} + \sum_{k=\chi_n}^{n-1} e^{-(n-1-k)\delta} (1 - U_k)^{\frac{1}{\alpha}} S_k \right) =: D_n$$
The Algorithm

1: Sample backwards in time \( \{(D_n, \Theta_n)\} \) until \(-\sigma\), the first time in which \( \{D_n \leq a(\Theta_n)\} \)
2: Put \( X_{-\sigma+1} = \psi(a(\Theta_{-\sigma}), \Theta_{-\sigma}) \)
3: Compute recursively \( X_n = \psi(X_{n-1}, \Theta_{n-1}) \)
4: return \( X_0 \) \( \triangleright \) Here \( X_0 \sim \bar{S}(\alpha, \rho) \)
Sanity Check $(\alpha, \beta) = (1.3, -1)$

- Average sampling-time for each r.v.: 0.011774 seconds
- Kolmogorov-Smirnov test $p$-value: 0.9213
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**Discretisation** \((\alpha, \beta) = (1.3, -1)\)

- Kolmogorov distances for \(N = 8,000\) and \(2,000\) are 0.253 and 0.174 respectively.

![Empirical CDF of 10000 samples for \(\alpha=1.3, \beta=-1\)](image-url)
The Dominating Process

Discretisation \((\alpha, \beta) = (1.3, 1)\)

- Kolmogorov distances for \(N = 8,000\) and \(2,000\) are \(0.125\) and \(0.088\) respectively.
References

