Exponential Families and Oriented Matroids

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Motivation

The present work [3] was motivated by the following two questions:

1. Which support sets occur in the closure of an exponential family?
2. What happens to the results of algebraic statistics in the case of non-algebraic exponential families?

Notation

$\mathcal{X}$ – the state space (finite set) of cardinality N
A – a (real valued) matrix of size $(d+1)$ by N, containing the constant row in its row span
$\mathcal{P}(\mathcal{X})$ – the probability simplex over $\mathcal{X}$

The *exponential family* $\mathcal{E}_A$ is the set of all probability measures of the form

$$P_\theta(x) = \frac{1}{Z_\theta} \exp\left(\sum_{i=0}^{d} \theta_i A_{i,x}\right).$$

Here $\theta \in \mathbb{R}^{d+1}$ is a vector of parameters, and Z_θ ensures normalization. If A contains only integer entries, then $\mathcal{E}_A$ is called *algebraic*.

Result 1: Implicit description

Theorem 1. *Let $\mathcal{E}_A$ be an exponential family. Its closure $\overline{\mathcal{E}_A}$ equals the set*

$$\{P \in \mathcal{P}(\mathcal{X}) : P^{u_+} = P^{u_-} \text{ for all } u = u_+ - u_- \in \ker A\}.$$

Here $P^v := \prod_{x\in\mathcal{X}} P(x)^{v(x)}$.

This has to be compared with:

Theorem (Geiger, Meek, Sturmfels [2]). *Let $\mathcal{E}_A$ be an algebraic exponential family. Then $\overline{\mathcal{E}_A}$ equals the intersection of the toric variety defined by the polynomials*

$$P^{u_+} = P^{u_-} \text{ for all } u = u_+ - u_- \in \ker_{\mathbb{Z}} A$$

with $\mathcal{P}(\mathcal{X})$.

In the algebraic case Hilbert's theorem ensures that a finite number of equations is enough (→Markov bases). This is always true:

Theorem 2. *Let $\mathcal{C}$ be a* circuit basis *of A. Then*

$$\overline{\mathcal{E}_A} = \{P \in \mathcal{P}(\mathcal{X}) : P^{u_+} = P^{u_-} \text{ for all } u = u_+ - u_- \in \mathcal{C}\}.$$

Result 2: Support sets

Theorem 3. *Let $S \subseteq \mathcal{X}$ be nonempty. Then there exists a probability measure $P \in \overline{\mathcal{E}_A}$ with support $\operatorname{supp}(P) = S$ if and only if the following holds for all signed circuits $(M, N) \in \mathcal{C}(A)$:*

$$M \subseteq S \quad \Leftrightarrow \quad N \subseteq S.$$

Result 3: Parametrization of the closure

We may also parametrize $\mathcal{E}_A$ by the "monomial parametrization"

$$P_\xi(x) = \frac{1}{Z_\xi} \prod_{i=0}^{d} \xi_i^{A_{i,x}},$$

where $\xi_i \in (0, \infty)$. If $A_{i,x} \geq 0$ for all x, then it is possible to parametrize a part of the boundary $\overline{\mathcal{E}_A} \setminus \mathcal{E}_A$ by allowing $\xi_i = 0$ (as long as $Z_\xi \neq 0$).

Theorem 4. *Let A' be a matrix the rows of which contain one positive cocircuit vector for every positive cocircuit of A. Then $\mathcal{E}_A = \mathcal{E}_{A'}$, and the image of the monomial parametrization of $\mathcal{E}_{A'}$ consists of $\overline{\mathcal{E}_{A'}} = \overline{\mathcal{E}_A}$.*

Oriented matroids

A signed subset X of $\mathcal{X}$ is a pair (X^+, X^-) of disjoint subsets of $\mathcal{X}$. Alternatively, X is a sign vector $X \in \{0, \pm1\}^{\mathcal{X}}$. The set $\underline{X} := X^+ \cup X^-$ denotes the *support* of X.

Let $\mathcal{C} \neq \emptyset$ be a collection of signed subsets of $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{C})$ is called an *oriented matroid* iff:

(C1) $\mathcal{C} = -\mathcal{C}$, (*symmetry*)

(C2) for all $X, Y \in \mathcal{C}$, if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$, (*incomparability*)

(C3) for all $X, Y \in \mathcal{C}$, $X \neq -Y$, and $e \in X^+ \cap Y^-$ there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$. (*weak elimination*)

Elements of $\mathcal{C}$ are called *signed circuits*.

Most important example ("representable matroids"):
Let A be a matrix. Then

$$\mathcal{C} = \left\{(\operatorname{supp}(n^+), \operatorname{supp}(n^-)) : n \in \ker A \text{ has inclusion minimal support}\right\}$$

determines an oriented matroid. Equivalently, $\mathcal{C}$ consists of the signed supports of the minimal linear relations among the columns of A.
A *circuit basis* is a set containing exactly one vector $v \in \ker A$ for each $X \in \mathcal{C}$ such that $\operatorname{sgn}(v) = X$.

The dual oriented matroid

To every oriented matroid corresponds a *dual oriented matroid*. Here we only explain the representable case (see [1]):
For every dual vector $l \in (\mathbb{R}^{d+1})^*$ let $N_l^+ := \{x \in \mathcal{X} : l(a_x) > 0\}$ and $N_l^- := \{x \in \mathcal{X} : l(a_x) < 0\}$. The signed subset $\operatorname{sgn}^*(l) := (N_l^+, N_l^-)$ is called a *covector*. A covector with inclusion minimal support is a *cocircuit*.
The set of cocircuits $\mathcal{C}^*$ forms an oriented matroid over $\mathcal{X}$, the *dual oriented matroid*.
It is representable by the following construction: Let A^* be a matrix such that the rows of A^* span the orthogonal complement of the rows of A. Then A^* "represents" $(\mathcal{X}, C^*)$.

Oriented matroids and polytopes

To a matrix A we can associate a polytope $\mathcal{M}_A$, the convex hull of its columns. Conversely, to each polytope we may associate the matrix of its vertices. Therefore each polytope has an oriented matroid.
$\Longrightarrow$ Many constructions generalize from polytopes to oriented matroids.

The face lattice of $\mathcal{M}_A$ corresponds to the dual oriented matroid: Faces are hyperplanes such that $\mathcal{M}_A$ is contained in one of the closed halfspaces. Thus facets (maximal faces) correspond to *positive cocircuits*.

It is known that the possible support sets of $\overline{\mathcal{E}_A}$ correspond to the faces of $\mathcal{M}_A$. Using this result, Theorem 3 is related to a general characterization of faces of "matroid polytopes" due to Las Vergnas[1].

Generalization

All three results generalize to non-uniform reference measures.

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, *Oriented matroids*, Cambridge University Press, 1993.

[2] D. Geiger, C. Meek, B. Sturmfels, *On the toric algebra of graphical models*, Ann. Stat. **34** (2006), no. 5, 1463–1492.

[3] J. Rauh, T. Kahle, N. Ay, *Support Sets in Exponential Families and Oriented Matroid Theory*, submitted to Int. J. Approx. Reas. (`arXiv:0906.5462`).