Shape Depth

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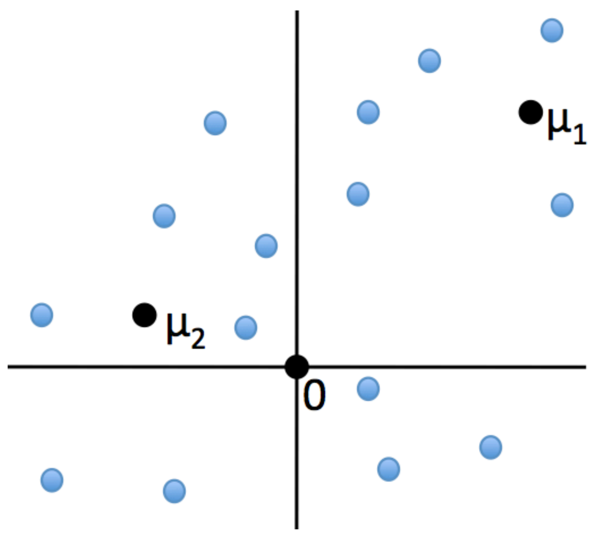
1. Tukey's halfspace depth and Mizera's tangent depth.



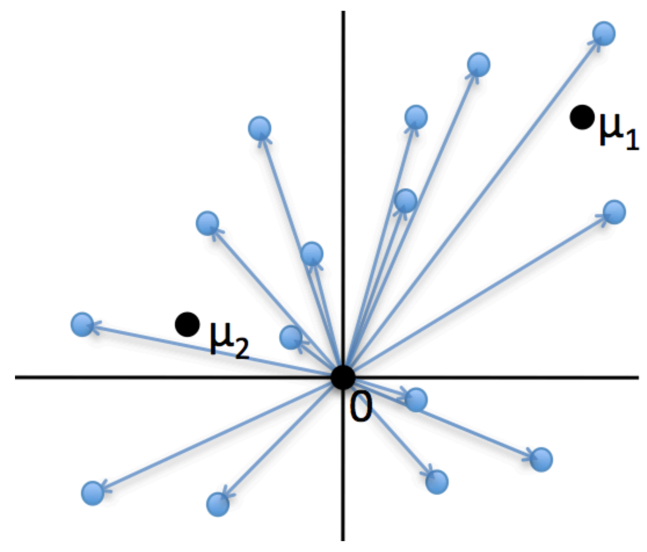
In the multivariate location setup, Tukey (1975) introduced the notion of halfspace median as $\mathrm{argmax}_{\boldsymbol{\mu}\in\mathbb{R}^p} HD(\boldsymbol{\mu}; F_{\mathbf{X}})$, the vector with maximum halfspace depth. Here,

$$HD(\boldsymbol{\mu}; F_{\mathbf{X}}) = \inf_{\mathbf{u}\in S^{p-1}} \mathbb{P}_{F_{\mathbf{X}}}\{\mathbf{u}'(\mathbf{X}-\boldsymbol{\mu}) \geq 0\}.$$

In the empirical setup, the halfspace depth of $\boldsymbol{\mu}$ can be seen as the smallest number of observations one may remove before $\boldsymbol{\mu}$ lies outside the convex hull of the remaining data.

Mizera (2002) generalized the concept of depth to any parametric model. Given a C^1 objective function $F : \Theta\times\mathcal{X} \to \mathbb{R}$, he defined the tangent depth of $\tilde{\boldsymbol{\theta}} \in \Theta$ as

$$TD\left(\tilde{\boldsymbol{\theta}}, F_{\mathbf{X}}\right) = HD\left(0_p; \nabla_{\boldsymbol{\theta}} F(\tilde{\boldsymbol{\theta}}, \mathbf{X})\right).$$



With respect to the construction of the maximum likelihood estimator, he suggested $F(\boldsymbol{\theta}, \mathbf{X}) = \log f_{\boldsymbol{\theta}}(\mathbf{X})$ as a candidate for the objective function.

2. The shape parameter.

In many classic multivariate problems, it is sufficient to know a *normalized* version of the dispersion/scatter matrix in order to perform the analysis (PCA, CCA, test of sphericity, etc.)

Definition

A scale functional is a (i) 1-*homogeneous* function $S : \mathcal{S}_p \to \mathbb{R}_0^+$ (i.e. $S(\lambda\boldsymbol{\Sigma}) = \lambda S(\boldsymbol{\Sigma})$) such that (ii) S is *differentiable*, with $\frac{\partial S}{\partial \Sigma_{11}}(\boldsymbol{\Sigma}) \neq 0$ for all $\boldsymbol{\Sigma} \in \mathcal{S}_p$ and (iii) satisfies $S(\mathbf{I}_p) = 1$.

The scale parameter associated with $\boldsymbol{\Sigma}$ is $\sigma_S^2 = S(\boldsymbol{\Sigma})$. The shape parameter is $\mathbf{V}_S = \boldsymbol{\Sigma}/S(\boldsymbol{\Sigma})$. Denote $\mathcal{V}_p^S = \{\mathbf{V} \in \mathcal{S}_p \,|\, S(\mathbf{V}) = 1\}$. Classical choices of scale functionals are $S_O(\boldsymbol{\Sigma}) = \boldsymbol{\Sigma}_{11}$, $S_T(\boldsymbol{\Sigma}) = \mathrm{tr}(\boldsymbol{\Sigma})/p$ and $S_D(\boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{1/p}$. We are interested in the *shape parameter* of p-variate elliptical distributions, with general density

$$\mathbf{x} \mapsto \frac{c_{p,g_1}}{|\boldsymbol{\Sigma}|^{1/2}} g_1\left(d(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\Sigma})\right) = \frac{\tilde{c}_{p,g}}{|\mathbf{V}_S|^{1/2}} g\left(d(\mathbf{x}, \boldsymbol{\theta}; \mathbf{V}_S)\right) = f_{\boldsymbol{\theta},\mathbf{V},g}(\mathbf{x}),$$

where $d(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\Sigma}) = \left((\mathbf{x}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\theta})\right)^{1/2}$, $\boldsymbol{\Sigma} \in \mathcal{S}_p$.

3. Shape depth.

Hallin and Paindaveine (2006) showed that the score is given by

$$\nabla_{\mathbf{V}} F(\mathbf{V}, \mathbf{X}) \propto \mathbf{M}_S^{\mathbf{V}} \left(\mathbf{V}^{\otimes 2}\right)^{-1/2} \mathbf{vec}\left(\varphi_{g_1}\!\left(\frac{d}{\sigma_S}\right)\frac{d}{\sigma_S}\mathbf{U}\mathbf{U}' - \mathbf{I}_p\right),$$

with $d = d(\mathbf{X}, \boldsymbol{\theta}; \mathbf{V})$, $\mathbf{U} = \mathbf{V}^{-1/2}(\mathbf{X}-\boldsymbol{\theta})/d$ and $\varphi_g = -g'/g$. $\mathbf{M}_S^{\mathbf{V}}$ is the $(p(p+1)/2-1)\times p$ matrix such that $(\mathbf{M}_S^{\mathbf{V}})'(\overset{\circ}{\mathbf{vech}})(\mathbf{v}) = \mathbf{vec}(\mathbf{v})$ for all matrix $\mathbf{v} \in \mathcal{S}_p$ such that $(\nabla S(\mathbf{vech}(\mathbf{V})))'(\mathbf{vech}(\mathbf{v})) = 0$. The unspecification of scale σ_S^2 leads to rather consider the efficient scores (obtained from projections along tangent spaces)

$$\nabla^*_{\mathbf{V}} F(\mathbf{V}, \mathbf{X}) \propto \mathbf{M}_S^{\mathbf{V}} \left(\mathbf{V}^{\otimes 2}\right)^{-1/2} \varphi_{g_1}\!\left(\frac{d}{\sigma_S}\right)\frac{d}{\sigma_S}\mathbf{vec}\left(\mathbf{U}\mathbf{U}' - \frac{1}{p}\mathbf{I}_p\right).$$

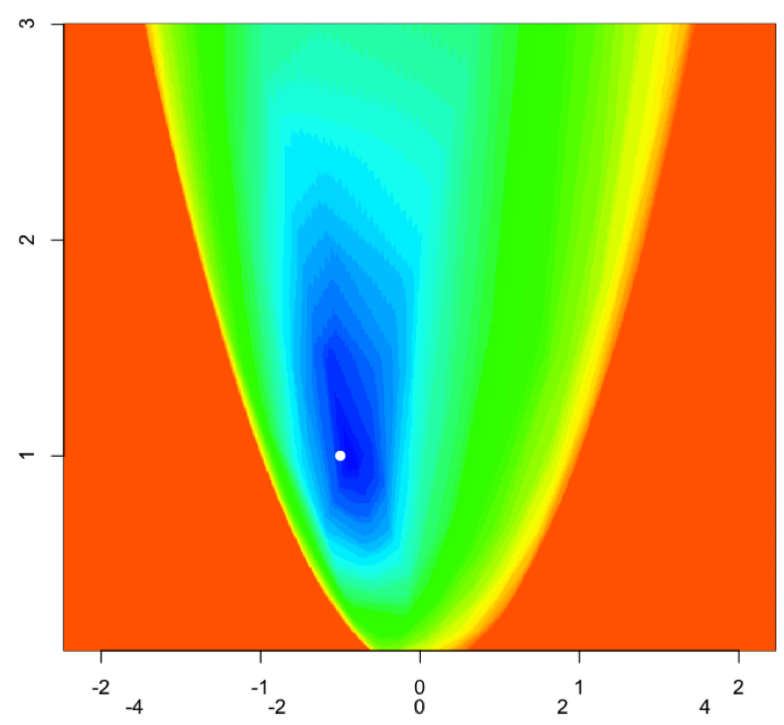
Definition

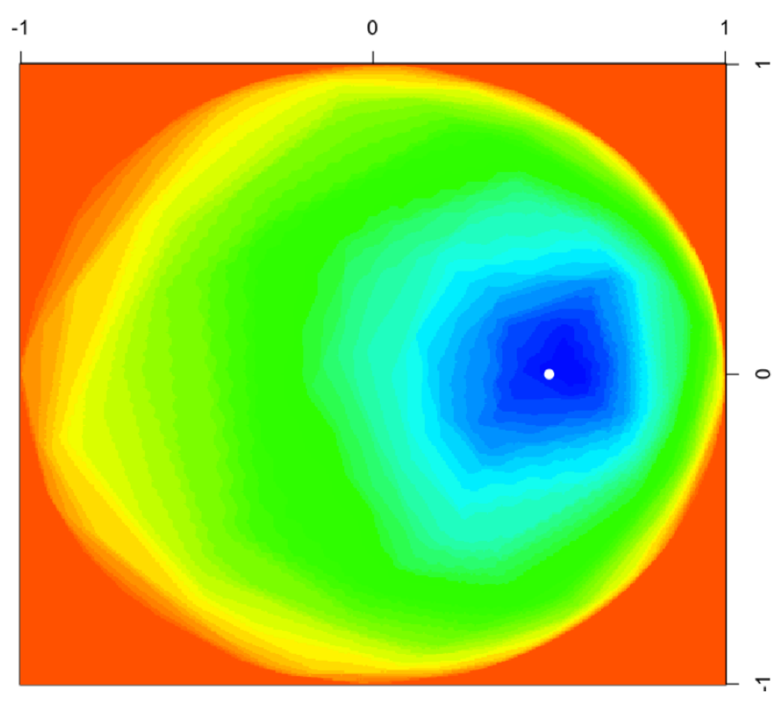
Let S be a scale functional, and $\mathbf{V} \in \mathcal{V}_p^S$. Define the vector $\mathbf{W} = \mathbf{M}_S^{\mathbf{V}_S} \left(\mathbf{V}^{\otimes 2}\right)^{-1/2} \mathbf{vec}\left(\mathbf{U}\mathbf{U}' - \frac{1}{p}\mathbf{I}_p\right)$. Then the *shape depth* of $\mathbf{V}$ is

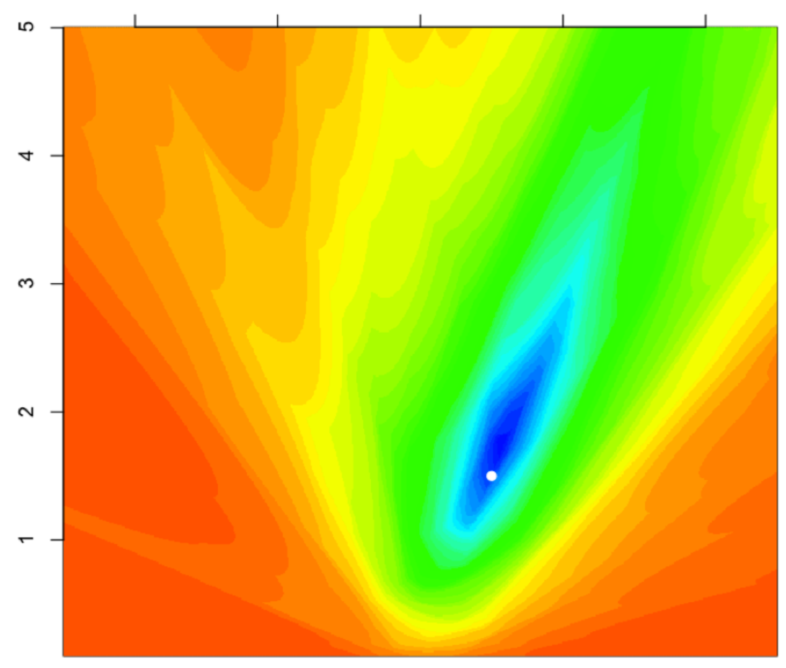
$$ShD(\mathbf{V}, S, F_{\mathbf{X}}) = HD\left(\mathbf{0}_p, \mathbf{W}\right).$$

Dimension 2 allows for representations since

- $\mathcal{V}_2^{S_O} \cong \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y > x^2\right\}$,
- $\mathcal{V}_2^{S_T} \cong \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < 1\right\}$ and
- $\mathcal{V}_2^{S_D} \cong \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y > 0\right\}$.







4. Some results.

Theorem 1

For any $\boldsymbol{\Sigma} \in \mathcal{S}_p$ and any scale functionals S_1 and S_2,

$$ShD\left(\frac{\boldsymbol{\Sigma}}{S_1(\boldsymbol{\Sigma})}, S_1, F_{\mathbf{X}}\right) = ShD\left(\frac{\boldsymbol{\Sigma}}{S_2(\boldsymbol{\Sigma})}, S_2, F_{\mathbf{X}}\right).$$

Theorem 2

Let S be a scale functional, $\mathbf{A}$ a non-singular $p \times p$ matrix and $\mathbf{V} \in \mathcal{V}_p^S$. Then, for all distributions $F_{\mathbf{X}}$,

$$ShD\left(\mathbf{V}, S, F_{\mathbf{X}}\right) = ShD\left(\frac{\mathbf{A}\mathbf{V}\mathbf{A}'}{S(\mathbf{A}\mathbf{V}\mathbf{A}')}, S, F_{\mathbf{A}\mathbf{X}}\right).$$

References

(1) Hallin, M. and Paindaveine, D. (2006). Semiparametrically efficient rank-based inference for shape I: Optimal rank-based tests for sphericity. *Annals of Statistics*, Vol. 34, pp. 2707-2756.

(2) Mizera, I. (2002). On depth and deep points: a calculus. *Annals of Statistics*, Vol. 30, No. 6, pp. 1681-1736.

(3) Tukey, J.W. (1975). Mathematics and the picturing of data. *Proceedings of the International Congress of Mathematicians* (Vancouver, B.C., 1974), Vol. 2, pp. 523-531.