Additional Mathematics Refresher Worksheet - Solutions

Department of Statistics, University of Warwick

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1 Core

Question 1.1 *For each of the following functions, domains and codomains, state whether it is:*

a) Injective (but not surjective)

b) Surjective (but not injective)

c) Bijective

d) Neither injective nor surjective

In the cases where the function is bijective, also find the inverse.

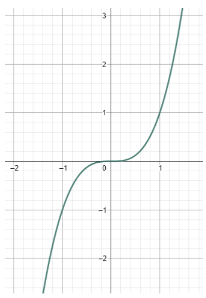
In the cases where the function is not injective, propose a new domain such that the function would be injective (if possible).

In the cases where the function is not surjective, propose a new codomain such that the function would be surjective (if possible).

1. $f(x) = 3x + 7;\ X = Y = \mathbb{R}$
2. $f(x) = x^2;\ X = Y = \mathbb{R}$
3. $f(x) = e^x;\ X = Y = \mathbb{R}$
4. $f(x) = \sqrt{x};\ X = \{x \in \mathbb{R} : x \geq 0\}, Y = \mathbb{R}$
5. $f(x) = x^3;\ X = Y = \mathbb{R}$
6. $f(x) = \sin(x);\ X = [0, 2\pi],\ Y = \mathbb{R}$
7. $f(x) = \frac{1}{x};\ X = \mathbb{R} \setminus \{0\}, Y = \mathbb{R}$
8. $f(x) = |x|; X = Y = \mathbb{R}$

Question 1.2 *For the function $f(x) = x^3; X = Y = \mathbb{R}$, write the new function $g(x)$ after the following transformations:*

1. *Translation by 1 to the right*
2. *Translation by 3 down*
3. *Reflection across the y-axis*
4. *Reflection across the x-axis*
5. *Stretched horizontally by a factor of 2*



Question 1.3 *For each of the following, find the first derivative:*

a) $y = x^{\frac{3}{2}}$

b) $y = \sin(5x^2)$

c) $y = e^{\tan(x)}$

d) $y = x^3 \log(3x)$

e) $y = \frac{\cos(x^2)}{\sqrt{x}}$

f) $y = \log(\sin(x))$

g) $y = \sin(\cos(x^2))$

Question 1.4 *Evaluate* $\int_0^1 x^3 + x - \sqrt{x}\mathrm{d}x$

Question 1.5 *Use partial fractions to evaluate* $\int_2^4 \frac{1}{x^2-1}\mathrm{d}x$

Question 1.6 *Use the indefinite integral to evaluate the improper integral* $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$

Question 1.7 *Solve the following differential equations,*

a) $y' = \dfrac{x^2}{y}$.

b) $y' = \dfrac{x^2}{y(1+x^3)}$.

c) $y' + y^2 \sin x = 0$.

d) $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$.

Question 1.8 *Write the general solution of*

$$\frac{d^2x}{dt^2} = -\omega^2 x\,, \qquad \omega \in \mathbb{R}\,.$$

Question 1.9 *In triangle* ABC, $\overrightarrow{AB} = 6\boldsymbol{i} + 2\boldsymbol{j}$ *and* $\overrightarrow{AC} = 8\boldsymbol{i} - 5\boldsymbol{j}$.

a) *Find the vector* $\overrightarrow{BC}$

b) *Find the length of the line* AB

Question 1.10 *Relative to the origin* O*:*

the point A *has position vector* $2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$

the point B *has position vector* $4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

and the point C *has position vector* $a\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$, *where* $a < 0$ *is a constant.*

D *is the point such that* $\overrightarrow{AB} = \overrightarrow{BD}$.

a) *Find the position vector of* D

b) *Given that* $\overrightarrow{AC} = 4$, *find the value of* a

Question 1.11 *With respect to a fixed origin* O, *the line* l_1 *is given by the equation*

$$\boldsymbol{r} = \begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

where μ is a scalar parameter. The point A lies on l_1 where $\mu = 1$.

a) Find the coordinates of A.

The point P has position vector

$$\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

The line l_2 passes through the point P and is parallel to the line l_1.

b) Write down a vector equation for the line l_2.

Question 1.12 *Let the following matrices be defined as*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -5 & 6 \\ 7 & 11 \\ -3 & -4 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 4 & 1 \end{bmatrix}.$$

In the following, compute the resultant value, or explain why it cannot be computed:

(a) $\boldsymbol{AB}$.

(b) $\boldsymbol{A} - \boldsymbol{D}$.

(c) $\boldsymbol{BA}$.

(d) $\boldsymbol{BC}$.

(e) $\boldsymbol{AD} + \boldsymbol{CE}$.

(f) $\boldsymbol{EB}^\top$.

(g) $\boldsymbol{A}^2$.

(h) $\boldsymbol{C}^2$.

(i) $|\boldsymbol{A}|$.

(j) $|\boldsymbol{B}|$.

(k) $\boldsymbol{A}^{-1}$.

(l) $\boldsymbol{D}^{-1}$.

Question 1.13 (2023 A Level Maths Paper 1 Q1) *Find the coefficient of x^7 in the expansion of $(2x-3)^7$.*

a) -2187

b) -128

c) 2

d) 128

Question 1.14 (2022 A Level Maths Paper 1 Q2) *A periodic sequence is defined by $U_n = (-1)^n$. State the period of the sequence.*

a) -1

b) -0

c) 1

d) 2

Question 1.15 (2021 A Level Maths Paper 1 Q1) *A geometric sequence has a sum to infinity of -3. A second sequence is formed by multiplying each term of the sequence by -2. What is the sum of the new sequence?*

a) The sum to infinity doesn't exist

b) -6

c) -3

d) 6

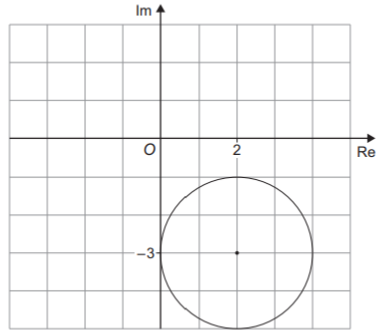
Question 1.16 (2021 A Level Maths Paper 1 Q6) *The ninth term of an arithmetic sequence is 3. The sum of the first n terms of the sequence is S_n, with $S_{21} = 42$.*

a) Find the first term and common difference of the series

A second arithmetic series has first term -18 and common difference $\frac{3}{4}$. The sum of the first n terms of the sequence is T_n.

b) Find the value of n such that $T_n = S_n$.

Question 1.17 (2023 AQA Further Maths Paper 1 Q2) *The diagram below shows a locus on an Argand diagram.*



Which of the equations below represents the locus shown above?

a) $|z - 2 + 3i| = 2$

b) $|z + 2 - 3i| = 2$

c) $|z - 2 + 3i| = 4$

d) $|z + 2 - 3i| = 4$

Question 1.18 (2023 AQA Further Maths Paper 1 Q6(a)) *Express $-5 - 5i$ in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$*

2 Further

Question 2.1 (2019 AQA A-Level Paper 1 Q6) *The function f is defined by*

$$f(x) = \tfrac{1}{2}(x^2 + 1), x \geq 0$$

a) Find the range of f

b) Find $f^{-1}(x)$

c) State the range of $f^{-1}(x)$

d) State the transformation which maps the graph $y = f(x)$ onto the graph $y = f^{-1}(x)$

e) Find the coordinates of the point of intersection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Question 2.2 (2022 TMUA Paper 1 Q1) *How many solutions are there to the equation*

$$2\cos^4(\theta) - 5\cos^2(\theta) + 3 = 0$$

Question 2.3 (2022 TMUA Paper 2 Q5) *A straight line L passes through $(1, 2)$. Let P be the statement: "**if** the y-intercept of L is negative, **then** the x-intercept of L is positive."*

*Which of the following statements **must** be true? (note: multiple can be true)*

a) P

b) The converse of P

c) The contrapositive of P

Question 2.4 (2018 Maths Paper 3 Q2) *A curve has equation $y = x^5 + 4x^3 + 7x + q$, where q is a positive constant. Find the gradient of the curve at the point $x = 0$.*

Question 2.5 (2024 Maths Paper 3 Q4) *A curve has equation $y = x^4 + 2^x$. Find an expression for $\frac{dy}{dx}$.*

Question 2.6 (2018 Maths Paper 1 Q5) *A curve is defined by the parametric equations*

$$x = \frac{4}{2^t} + 3$$
$$y = 3 \times 2^t - 5$$

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4} \times 2^{2t}$

Question 2.7 *Solve* $\int e^x \cos(x)\mathrm{d}x$

Question 2.8 (2024 AQA Maths, Paper 1 Q18) *Use a suitable substitution to show that* $\int_0^4 (4x+1)\sqrt{2x+1}\mathrm{d}x$ *can be written as* $\frac{1}{2}\int_a^9 (2u^{\frac{3}{2}} - u^{\frac{1}{2}})\mathrm{d}u$, *where* a *is a constant to be found.*

Hence, or otherwise, show that:

$$\int_0^4 (4x+1)\sqrt{2x+1}\mathrm{d}x = \frac{1322}{15}$$

Question 2.9 *Naively, "evaluate"* $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$ *by finding the indefinite integral and plugging in the bounds. Draw a graph to convince yourself that this answer is in fact wrong, and use improper integration to find the correct answer.*

Question 2.10 (AQA Specimen Paper 1, Q8b) *Find the exact value of* $\int_0^1 2^x\sqrt{3+2^x}\mathrm{d}x$. *Fully justify your answer.*
Hint: use the previous question and a substitution.

Question 2.11 (June 2024 — Maths Paper 1 Q20)
A gardener stores rainwater in a cylindrical container. The container has a height of 130 centimetres. The gardener empties the water from the container through a hose. The hose is attached 5 centimetres from the bottom of the container. At time t *minutes after the hose is switched on, the depth of water,* h *centimetres, in the container decreases at a rate which is proportional to* $h - 5$. *Initially the container of water is full, and the depth of water is decreasing at a rate of 1.5 centimetres per minute*

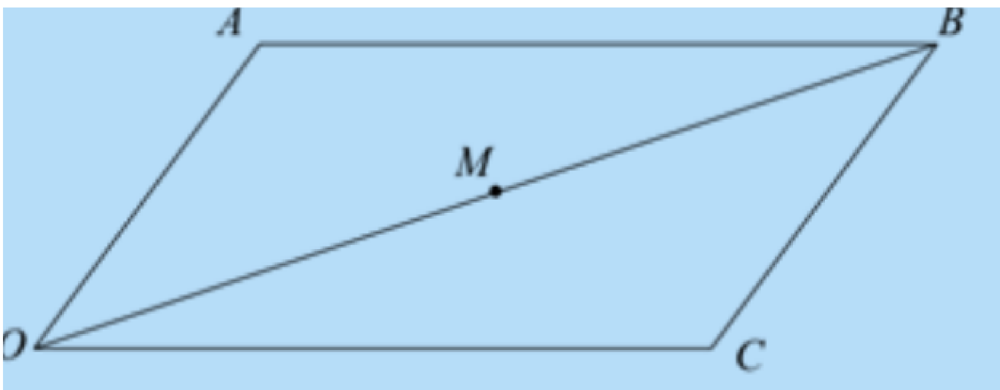
(a) Show that

$$\frac{dh}{dt} = -0.012(h-5)$$

(b) Solve the equation to find an expression of h *in terms of* t.

(c) Find the time taken for the container to be half empty. Give your answer to the nearest minute.

Question 2.12 $OABC$ *is a parallelogram with* $\overrightarrow{OA} = \boldsymbol{a}$ *and* $\overrightarrow{OC} = \boldsymbol{c}$. M *is the midpoint of* $\overrightarrow{OB}$.



a) Find, in terms of $\boldsymbol{a}$ *and* $\boldsymbol{c}$, *simplifying your answers:*

i) $\overrightarrow{AC}$

ii) $\overrightarrow{OM}$

b) Hence prove the diagonals of a parallelogram bisect one another

Question 2.13 *With respect to a fixed origin O, the lines l_1 and l_2 are given by:*

$$l_1 : \boldsymbol{r} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}$$

and

$$l_2 : \boldsymbol{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$$

where λ and μ are scalar parameters. The lines intersect at the point X.

(a) *Find the coordinates of X*

(b) *Find the size of the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places.*

The point A lies on l_1 and has position vector:

$$\begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix}$$

(c) *Find the distance AX, giving your answer as a surd in its simplest form.*

The point Y lies on l_2. Given that the vector $\overrightarrow{YA}$ is perpendicular to l_1:

(d) *find the distance of YA, giving your answer to one decimal place.*

The point B lies on l_1 where $|\overrightarrow{AX}| = 2|\overrightarrow{AB}|$.

(e) *Find the two possible position vectors of B.*

Question 2.14 (2023 Further Maths Paper 2 Q8) *Let $\boldsymbol{A}$ be a non-singular 2×2 matrix with $\boldsymbol{A}^\top$ being the transpose of $\boldsymbol{A}$.*

(a) *Using the result*

$$(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top,$$

show that

$$(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}.$$

(b) *It is given that $\boldsymbol{A} = \begin{bmatrix} 4 & 5 \\ 1 & k \end{bmatrix}$, where k is a real constant.*

(b) (i) *Find $(\boldsymbol{A}^{-1})^\top$ in terms of k.*

(b) (ii) *What is the restriction on possible values of k?*

Question 2.15 (2023 A Level Maths Paper 1 Q11) *The n^{th} term of a sequences is u_n, with the sequence defined by*

$$u_{n+1} = pu_n + 70$$

where $u_1 = 400$ and p is constant.

a) *Find an expression in terms of p for u_2*

b) *It is given that $u_3 = 382$. Show that p satisfies the equation*

$$200p^2 + 35p - 156 = 0$$

c) *Given that the sequence is a decreasing sequence, find the value of u_4 and the value of u_5*

d) *The limit of u_n as n tends towards infinity is L. Write down an equation for L*

e) *Find the value of L*

Question 2.16 (2022 A Level Maths Paper 1 Q9) *The first three terms of an arithmetic sequence are given by:*

$$2x + 5 \qquad 5x + 1 \qquad 6x + 7$$

a) Show that $x = 5$ *is the only value which gives an arithmetic sequence*

b) Write down the value of the first term of the sequence

c) Find the common difference of the sequence

d) Find N *such that the sum of the first* N *terms of the arithmetic sequence is* S_N *and*

$$S_N < 100,000$$
$$S_{N+1} > 100,000$$

Question 2.17 (2022 A Level Maths Paper 1 Q12) *A geometric sequence has first term* 1 *and common ratio* $\frac{1}{2}$.

a) Find the sum to infinity of the sequence

b) Hence, or otherwise, evaluate

$$\sum_{n=}^{\infty} (\sin 30^o)^n$$

Question 2.18 (2021 A Level Further Maths Paper 1 Q1) *Find*

$$\sum_{r=1}^{20} r^2 - 2r$$

from the below answers:

a) 2,450

b) 2,660

c) 5,320

d) 43,680

Question 2.19 (AQA 2019 FM Paper 1 Q4) *Solve the equation* $2z - 5iz^* = 12$

Question 2.20 (AQA 2019 FM Paper 1 Q8) *a) If* $z = cos\theta + isin\theta$ *, use de Moivre's theorem to prove that*

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

b) Express $\sin^5 \theta$ *in terms of* $\sin 5\theta, \sin 3\theta$ *and* $\sin \theta$
c) Hence show that:

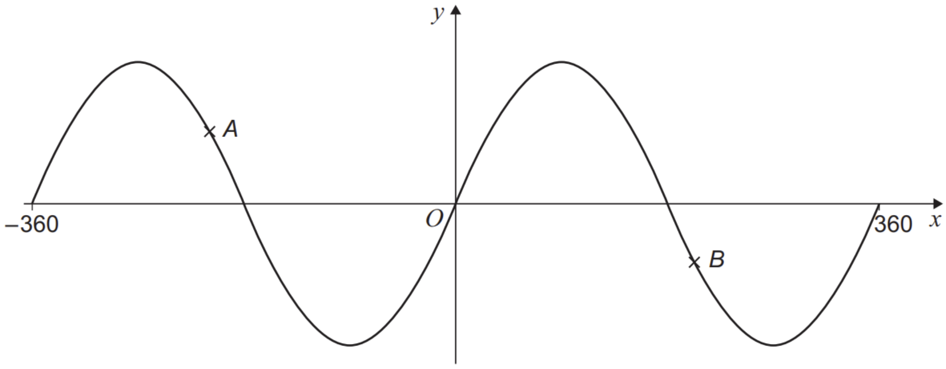
$$\int_0^{\frac{\pi}{3}} \sin^5(\theta) d\theta = \frac{53}{480}$$

3 Advanced

Question 3.1 (2023 AQA A-Level Paper 1 Q10) *The curve with equation*

$$y = \sin(x)^o$$

for $-360 \leq x \leq 360$ *is shown below*



Point A on the curve has coordinates $(a, 0.5)$.

a) *Find the value of a*

b) *State the value of* $\sin(180^o - a^o)$

Point B on the curve has coordinates $(b, -\frac{3}{7})$.

c) *Find the exact value of* $\sin(b^o - 180^o)$

d) *Find the exact value of* $\cos(b^o)$

Question 3.2 (2023 AQA Further Maths A-Level Paper 1 Q7) *The function f is defined by:*

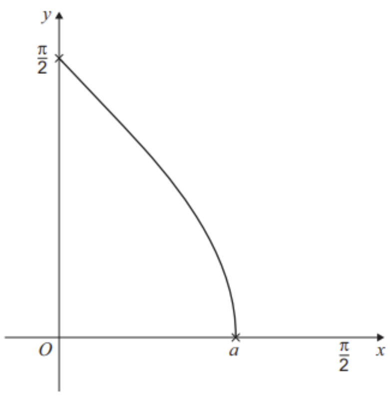
$$f(x) = |\sin(x) + \tfrac{1}{2}|, 0 \leq x \leq 2\pi$$

Find the set of values of x for which $f(x) \geq \frac{1}{2}$

Question 3.3 (2023 AQA A-Level Paper 1 Q13) *The function f is defined by:*

$$f(x) = \arccos(x),\ \ 0 \leq x \leq a$$

The curve with equation $y = f(x)$ is seen below:



a) *Find the value of a*

b) *On the diagram above, sketch the curve of $y = \cos(x)$ **and** the line $y = x$ both for $0 \leq x \leq \frac{\pi}{2}$*

c) *Explain why the solution to the equation $x - \cos(x) = 0$ must also be a solution to the equation $\cos(x) = \arccos(x)$*

Question 3.4 (2019 Maths Paper 1 Q10) *The volume of a spherical bubble is increasing at a constant rate. Show that the rate of increase of the radius, r, of the bubble is inversely proportional to r^2. The volume of a sphere is $\frac{4}{3}\pi r^3$*

Question 3.5 (TMUA 2023 Paper 1 Q11) *It is given that $f(x) = x^2 - 6x$. The curves $y = f(kx)$ and $y = f(x - c)$ have the same minimum point, where $k > 0$ and $c > 0$. Find an expression for k in terms of c.*

Question 3.6 (2019 Further Maths Paper 1 Q2) *The first two nonzero terms of the Maclaurin series expansion of $f(x)$ are x and $-\frac{1}{2}x^3$. Which one of the following could be $f(x)$?*

$$xe^{\frac{1}{2}x^2} \qquad \frac{1}{2}\sin(2x) \qquad x\cos(x) \qquad (1+x^3)^{-\frac{1}{2}} \tag{1}$$

Question 3.7 *Solve* $\int \sin^3(x)\mathrm{d}x$

Question 3.8 *June 2024, AQA Further Maths, Paper 1 Q11: Find* $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\arctan(x)\right)$ *and hence find* $\int 2x\arctan(x)\mathrm{d}x$.

Question 3.9 (2023 Further Maths Paper 1 Q15) *Find the general solution of the differential equation,*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = \cos 2x + 5x$$

Question 3.10 *An octopus is able to catch any fish that swim within a distance of 2m from the octopus's position.*
A fish F swims for point A to point B. The octopus is modelled as a fixed particle at the origin O.
Fish F is modelled as a particle moving in a straight line from A to B.
Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

a) *Use the model to determine whether one or not the octopus is able to catch fish F*
b) *Criticise the model in relation to fish F*
c) *Criticise the model in relation to the octopus*

Question 3.11 (2023 TMUA Paper 1 Q4) *Evaluate*

$$\sum_{n=0}^{\infty} \frac{\sin(n\pi + \frac{\pi}{3})}{2^n}$$

a) *0*
b) $\frac{1}{3}$
c) $\frac{\sqrt{3}}{3}$
d) $\sqrt{3}$
e) *3*

Question 3.12 (2023 TMUA Paper 2 Q16) *A sequence is defined by*

$$u_1 = a$$
$$u_2 = b$$
$$u_{n+2} = u_n + u_{n+1}$$

for $n \geq 1$ and where a and b are positive integers. The highest common factor of a and b is 7.
*Which of the following **must** be true.*

I) *u_{2023} is a multiple of 7*
II) *If u_1 is not a factor of u_2, then u_1 is not a factor of u_n for any n ¿ 1*
III) *The highest common factor of u_1 and u_5 is 7*

a) *none of them*
b) *I only*
c) *II only*
d) *III only*
e) *I and II only*
f) *I and III only*

g) II and III only

h) I, II and III

Question 3.13 (2022 TMUA Paper 1 Q5) *The terms x_n of a sequence follow the rule:*

$$x_{n+1} = \frac{x_n + p}{x_n + q}$$

where $p, q \in \mathbb{R}$. Given that $x_1 = 3, x_2 = 5$ and $x_3 = 7$, find the value of x_4.

A) -5

B) 5

C) $\frac{51}{7}$

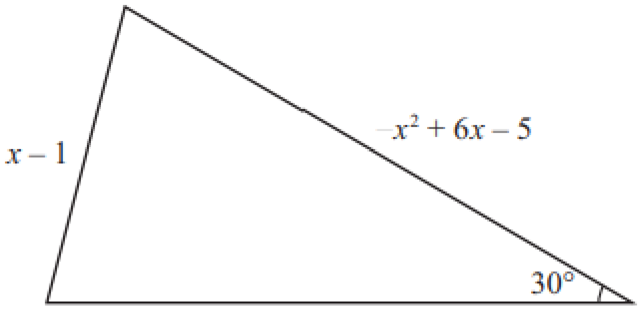
D) $\frac{15}{2}$

E) $\frac{23}{3}$

F) 9

G) 11

H) 13



Question 3.14
Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A) $1 < x < 3$

B) $1 < x < 4$

C) $1 < x < 5$

D) $3 < x < 4$

E) $3 < x < 5$

F) $4 < x < 5$

Note: the sides are $x - 1$ and $-x^2 + 6x - 5$: the minus signs can be difficult to see.