Additional Mathematics Refresher Worksheet - Solutions

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1 Core

Question 1.1 *For each of the following functions, domains and codomains, state whether it is:*

a) Injective (but not surjective)

b) Surjective (but not injective)

c) Bijective

d) Neither injective nor surjective

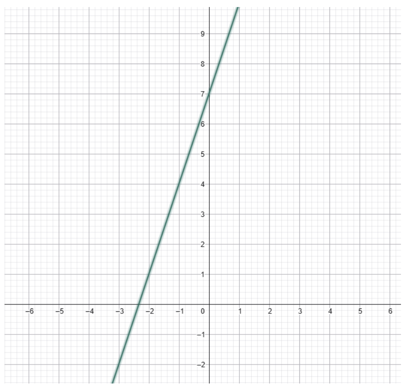
In the cases where the function is bijective, also find the inverse.

In the cases where the function is not injective, propose a new domain such that the function would be injective (if possible).

In the cases where the function is not surjective, propose a new codomain such that the function would be surjective (if possible).

1. $f(x) = 3x + 7;\ X = Y = \mathbb{R}$
2. $f(x) = x^2;\ X = Y = \mathbb{R}$
3. $f(x) = e^x;\ X = Y = \mathbb{R}$
4. $f(x) = \sqrt{x};\ X = \{x \in \mathbb{R} : x \geq 0\}, Y = \mathbb{R}$
5. $f(x) = x^3;\ X = Y = \mathbb{R}$
6. $f(x) = \sin(x);\ X = [0, 2\pi],\ Y = \mathbb{R}$
7. $f(x) = \frac{1}{x};\ X = \mathbb{R} \setminus \{0\}, Y = \mathbb{R}$
8. $f(x) = |x|; X = Y = \mathbb{R}$

Solution 1.1 *1.* $f(x) = 3x + 7;\ X = Y = \mathbb{R}$
A graph shows us that f is injective; there is no horizontal line that would cut the graph in two places.



It is also surjective, as every y value has some x value that will map to it; the line covers the entire of $\mathbb{R}$ without any gaps. We can show this by calculating the inverse (given it is bijective):

$$y = 3x + 7 \Rightarrow x = \frac{y-7}{3}$$

Hence $g(y) = f^{-1}(y) = \frac{y-7}{3}$.

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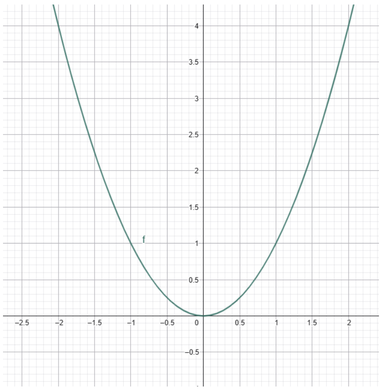
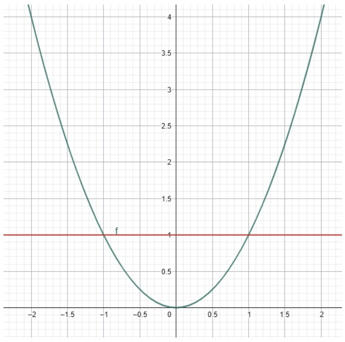
2. *$f(x) = x^2$; $X = Y = \mathbb{R}$.*
We start with a graph of the curve.



As the graph has a turning point, we can see that drawing a horizontal line at any positive y will cut the graph in two places. This makes sense as $(-x)^2 = x^2$. Hence, the function is not injective. If we removed the negative values of x such that $X = \{x \in \mathbb{R} : x \geq 0\}$, then the function would become injective.



Also, we know that $x^2 \geq 0$ and so $f(x)$ can never reach the negative values in $Y = \mathbb{R}$, and so f is not surjective. If we made $Y = \{y \in \mathbb{R} : y \geq 0\}$, then the function would be surjective.

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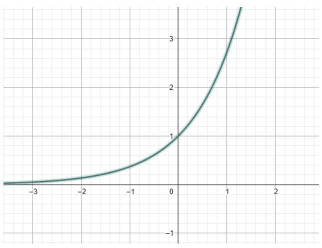
3. *$f(x) = e^x$; $X = Y = \mathbb{R}$ We start by drawing a graph once again.*



No horizontal line will cut the graph twice, and so the function is injective. However, e^x is strictly positive and so cannot reach the negative elements of Y. As with x^2, making $Y = \{y \in \mathbb{R} : y \geq 0\}$ would let f be surjective.

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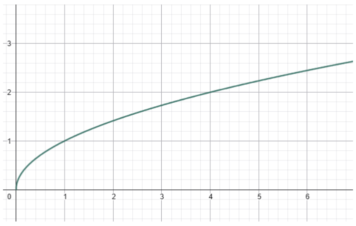
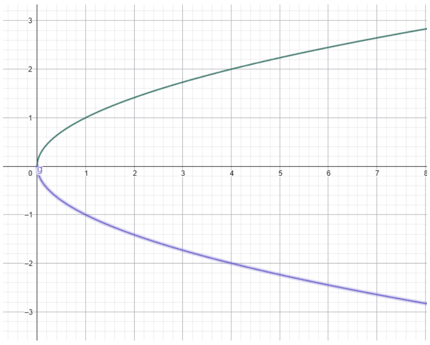
4. *$f(x) = \sqrt{x}$; $X = \{x \in \mathbb{R} : x \geq 0\}, Y = \mathbb{R}$*



The function is injective as no horizontal line will cut twice. However, once again, we cannot reach the negative values of Y to be surjective.

Nonetheless, the argument could be made that $\sqrt{x}$ can be both positive and negative, as both numbers squared will be the same. This would look like the below:



This would be surjective but would no longer be a function, as a vertical line will cut the graph in two places.

Notice how $\sqrt{x}$ is actually the inverse of x^2, and we showed how x^2 is not injective unless we restrict to either positive or negative x. This is exactly what we are doing for $\sqrt{x}$ in order to make it a function.

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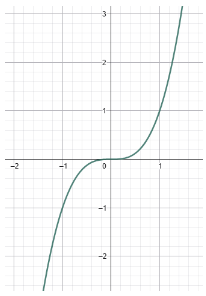
5. $f(x) = x^3;\ X = Y = \mathbb{R}$



This function is bijective; there is no turning point to cause issues with horizontal lines and it covers all of $\mathbb{R}$. The inverse is $g(y) = f^{-1}(y) = y^{\frac{1}{3}}$.

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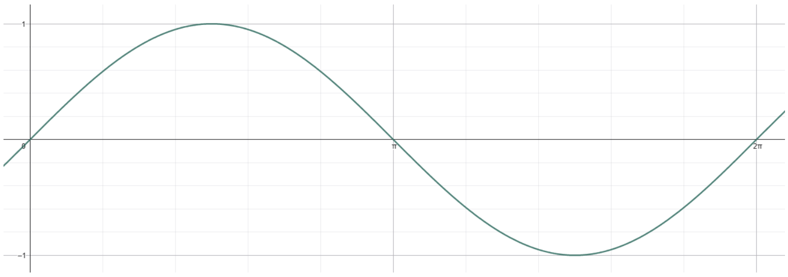
6. $f(x) = \sin(x);\ X = [0, 2\pi],\ Y = \mathbb{R}$



Clearly, a horizontal line anywhere between -1 and 1 will cut the graph in two places, so it is not injective. Changing $X = [\frac{\pi}{2}, \frac{3\pi}{2}]$ would remedy this.

As $-1 \leq \sin(x) \leq 1$, the function is not surjective. We change $Y = [-1, 1]$.

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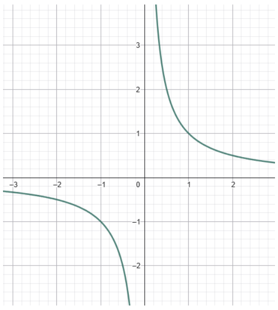
7. $f(x) = \frac{1}{x};\ X = \mathbb{R} \setminus \{0\}, Y = \mathbb{R}$



This is bijective; the asumptote at $x = 0$ has no effect. To calculate the inverse:

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

and so $g(y) = f^{-1}(y) = \frac{1}{y}$

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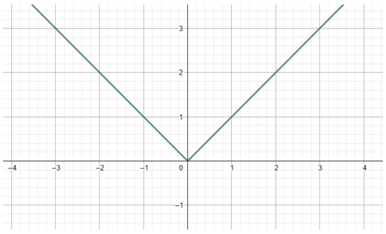
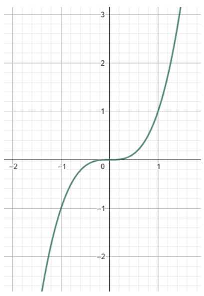
8. $f(x) = |x|; X = Y = \mathbb{R}$



This is is similar to x^2 in that it is neither injective nor surjective, but by making X and Y positive it would be bijective.

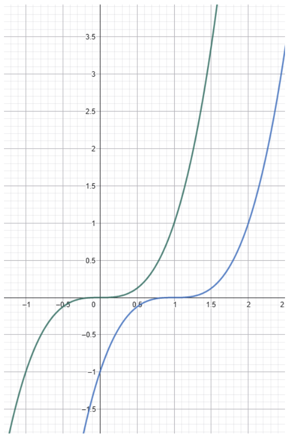
Question 1.2 *For the function $f(x) = x^3; X = Y = \mathbb{R}$, write the new function $g(x)$ after the following transformations:*

1. *Translation by 1 to the right*
2. *Translation by 3 down*
3. *Reflection across the y-axis*
4. *Reflection across the x-axis*
5. *Stretched horizontally by a factor of 2*



Solution 1.2 ***1.*** *Translation by 1 to the right.*

As we are translating across we need to change x. When we move in a positive x-direction, we subtract from x, and so $g(x) = (x - 1)^2$



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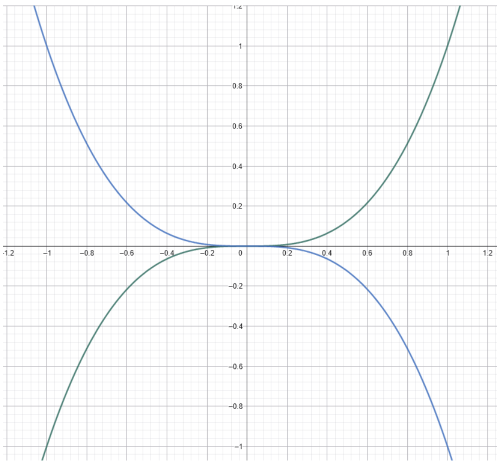
2. *Translation by 3 down.*

We are now moving y *rather than* x*, and so we subtract* 3 *from the* $y-$*value, which is* $f(x)$*. So* $g(x) = x^3 - 3$



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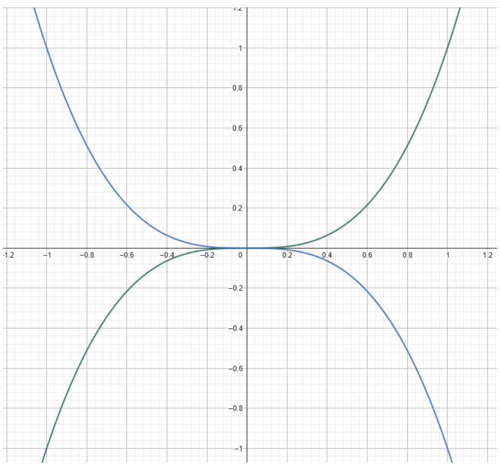
3. *Reflection across the y-axis*

Reflection across the y-axis equates to changing x *to* $-x$*. That is:*

$$y = (-x)^3 \Rightarrow y = -x^3$$

Hence, $g(x) = -x^3$



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4. Reflection across the x-axis

Similar to above but we change the sign of y rather than x. This gives $g(x) = -(x^3) = -x^3$, same as for reflecting across the $y-$axis.

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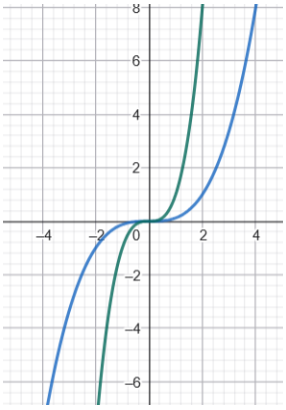
5. *Stretched horizontally by a factor of 2*
When you scale by a number greater than 1, the graph is compressed. When you scale by a number between 0 and 1, the graph is stretched. Thus, to stretch by a factor of 2, we must scale by $\frac{1}{2}$. That is: $g(x) = (\frac{1}{2}x)^3 = \frac{x^3}{8}$



Question 1.3 *For each of the following, find the first derivative:*

a) $y = x^{\frac{3}{2}}$

b) $y = \sin(5x^2)$

c) $y = e^{\tan(x)}$

d) $y = x^3 \log(3x)$

e) $y = \frac{\cos(x^2)}{\sqrt{x}}$

f) $y = \log(\sin(x))$

g) $y = \sin(\cos(x^2))$

Solution 1.3 *a)*

$$y = x^{\frac{3}{2}}$$

Use the formula:

$$y = x^a \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = ax^{a-1}$$

and so:

$$\begin{aligned} f'(x) &= \tfrac{3}{2}x^{\frac{3}{2}-1} \\ &= \tfrac{3}{2}x^{\frac{1}{2}} \end{aligned}$$

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b)

$$f(x) = \sin(5x^2)$$

We know the identity:

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = \cos(x)$$

As the argument is $5x^2$, we need to use the chain rule. Let $u = 5x^2$ with $y = \sin(u)$, and so:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 10x$$

Then,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \cos(u)10x \\ &= 10x\cos(5x^2)\end{aligned}$$

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c)

$$y = e^{\tan(x)}$$

Once again use the chain rule with $u = \tan(x)$ and $y = e^u$. We know $\frac{\mathrm{d}y}{\mathrm{d}u} = e^u$ and

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2(x)$$

Thus

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= e^u\sec^2(x) \\ &= e^{\tan(x)}\sec^2(x)\end{aligned}$$

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d)

$$y = x^3\log(3x)$$

For this we will need the product rule. Let $u = x^3$ and $v = \log(3x)$. It's easy to use a table for this:

$$\begin{array}{c|c} u = x^3 & v = \log(3x) \\ \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 & \frac{\mathrm{d}v}{\mathrm{d}x} = ? \end{array}$$

The cell for $\frac{\mathrm{d}v}{\mathrm{d}x}$ is missing because this derivative also requires the chain rule. If we let $w = 3x$ then the chain rule will give:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{w}3 = \frac{3}{3x} = \frac{1}{x}$$

Considering that $\log(3x) = \log(3) + \log(x)$, then this derivative makes sense. So, our table becomes:

$$\begin{array}{c|c} u = x^3 & v = \log(3x) \\ \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 & \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x} \end{array}$$

giving:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3}{x} + 3x^2\log(3x) = x^2(1 + 3\log(3x))$$

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e)

$$y = \frac{\cos(x^2)}{\sqrt{x}}$$

We could use the product rule with $u = \cos(x^2)$ and $v = x^{-\frac{1}{2}}$. Instead, we will use the quotient rule with $v = x^{\frac{1}{2}}$, but you could try both ways and show they are equal. We create the table once again:

$$\begin{array}{c|c} u = \cos(x^2) & v = x^{\frac{1}{2}} \\ \frac{\mathrm{d}u}{\mathrm{d}x} = ? & \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}} \end{array}$$

We need to use the chain rule for $u = \cos(x^2)$. Let $w = x^2$ and we get:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2x\sin(x^2)$$

This gives the table:

$$\begin{array}{r|r} u = \cos(x^2) & v = x^{\frac{1}{2}} \\ \frac{\mathrm{d}u}{\mathrm{d}x} = -2x\sin(x^2) & \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}} \end{array}$$

Then, the quotient rule gives our answer as:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^{\frac{1}{2}}(-2x\sin(x^2)) - \cos(x^2)\frac{1}{2}x^{-\frac{1}{2}}}{(x^{\frac{1}{2}})^2} \\ &= \frac{-2x^{\frac{3}{2}}\sin(x^2) - \frac{1}{2}x^{-\frac{1}{2}}\cos(x^2)}{x} \\ &= -2\sqrt{x}\sin(x^2) - \frac{1}{2}x^{-\frac{3}{2}}\cos(x^2)\end{aligned}$$

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f)

$$y = \log(\sin(x))$$

We use the chain rule again, with $u = \sin(x)$ and $y = \log(u)$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos(x)$ and $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u}$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sin(x)}\cos(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

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g)

$$y = \sin(\cos(x^2))$$

We use the chain rule except with 3 parts. Let $u = x^2$, let $v = \cos(u)$, and let $y = \sin(v)$. Then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

and so:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos(v)(-\sin(u))(2x) \\ &= -2x\cos(\cos(x^2))\sin(x^2)\end{aligned}$$

Question 1.4 *Evaluate $\int_0^1 x^3 + x - \sqrt{x}\mathrm{d}x$*

Solution 1.4 *This is a test of your ability to use linearity of the integral, turn roots into fractional indices, and use the formula $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$.*

$$\begin{aligned}\int_0^1 x^3 + x - \sqrt{x}\mathrm{d}x &= \int_0^1 x^3\mathrm{d}x + \int_0^1 x^1\mathrm{d}x - \int_0^1 x^{\frac{1}{2}}\mathrm{d}x \\ &= \frac{1}{4}x^4|_0^1 + \frac{1}{2}x^2|_0^1 - \frac{2}{3}x^{\frac{3}{2}}|_0^1 \\ &= \frac{1}{4} + \frac{1}{2} - \frac{2}{3}\end{aligned}$$

Hereafter, we will omit the handling of each integral term separately.

Question 1.5 *Use partial fractions to evaluate $\int_2^4 \frac{1}{x^2-1}\mathrm{d}x$*

Solution 1.5 *Noticing the difference of squares $x^2 - 1 = (x+1)(x-1)$ is key, as this allows us to decompose the fraction into two terms with linear denominators $(x+1)$ and $(x-1)$.*

$$\begin{aligned}
\frac{1}{x^2-1} &= \frac{A}{x+1} + \frac{B}{x-1} \\
&= \frac{Ax - A + Bx + B}{x^2-1} \\
&= \frac{(A+B)x - A + B}{x^2-1} \\
&\iff A+B = 0 \quad \& \quad -A+B = 0 \\
&\iff A = \frac{1}{2} \quad \& \quad B = -\frac{1}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_2^4 \frac{1}{x^2-1}\mathrm{d}x &= \frac{1}{2}\int_2^4 \frac{1}{x-1} - \frac{1}{x+1}\mathrm{d}x \\
&= \frac{1}{2}\log(x-1)|_2^4 - \frac{1}{2}\log(x+1)|_2^4 \\
&= \frac{1}{2}[\log(3) - \cancel{\log(1)} - \log(5) + \log(3)] \\
&= \frac{1}{2}[\log(3^2) - \log(5)] \\
&= \frac{1}{2}\log\left(\frac{9}{5}\right)
\end{aligned}$$

Question 1.6 *Use the indefinite integral to evaluate the improper integral $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$*

Solution 1.6 *By definition of the improper integral, $\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{n\to\infty}\int_1^n x^{-2}\mathrm{d}x$, so you should solve the integral with some n and then send $n \to \infty$.*

$$\int_1^n x^{-2}\mathrm{d}x = -x^{-1}|_1^n = -n^{-1} + 1$$

$$\lim_{n\to\infty} -\frac{1}{n} + 1 = 1$$

Question 1.7 *Solve the following differential equations,*

a) $y' = \dfrac{x^2}{y}$.

b) $y' = \dfrac{x^2}{y(1+x^3)}$.

c) $y' + y^2 \sin x = 0$.

d) $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$.

Solution 1.7 *a) The equation is separable, therefore we can write it as,*

$$y\frac{dy}{dx} = x^2\,,$$

and then integrate with respect to x,

$$\int y\frac{dy}{dx}dx = \int x^2 dx\,,$$

where, on the left-hand-side (LHS), we can change the integration variable to y,

$$\int y dy = \int x^2 dx\,.$$

Then follows that,

$$\frac{y^2}{2} = \frac{x^3}{3} + c_1\,, \qquad \textit{for some } c_1 \in \mathbb{R}\,.$$

We can then solve for y to obtain the general solution,

$$y(x) = \pm\sqrt{c_2 + \frac{2}{3}x^3}\,, \qquad \textit{for some } c_2 \in \mathbb{R}\,.$$

Which can be easily verified,

$$y' = \frac{(2x^3/3)'}{\pm 2\sqrt{c_2 + \frac{2}{3}x^3}} = \frac{x^2}{\pm\sqrt{c_2 + \frac{2}{3}x^3}} = \frac{x^2}{y}\,.$$

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b) The equation is again separable, so we can write it as,

$$y\frac{dy}{dx} = \frac{x^2}{1+x^2}\,.$$

And then we integrate with respect to x and make the change of variable on the LHS:

$$\int y dy = \int \frac{x^2}{1+x^2} dx\,.$$

Using that,

$$\int \frac{x^2}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan x\,,$$

it follows that,

$$\frac{y^2}{2} = x - \arctan x + c_1\,, \qquad \textit{for some } c_1 \in \mathbb{R}\,.$$

Therefore the general solution is,

$$y(x) = \pm\sqrt{c_2 + 2x - 2\arctan x}\,, \qquad \textit{for some } c_2 \in \mathbb{R}\,.$$

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c) The equation is separable too, so we can write it as,

$$\frac{1}{y^2}\frac{dy}{dx} = -\sin x\,.$$

We integrate,

$$\int \frac{1}{y^2} dy = -\int \sin x dx \quad \rightarrow \quad -\frac{1}{y} = \cos x + c_1 \qquad \textit{for some } c_1 \in \mathbb{R}\,.$$

Therefore, the solution is:

$$y(x) = \frac{1}{c_2 - \cos x}\,, \qquad \textit{for some } c_2 \in \mathbb{R}\,.$$

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d) The equation is separable too, so we can write it as,

$$(y + e^y)\frac{dy}{dx} = x - e^x\,.$$

We integrate,

$$\int (y + e^y) dy = \int (x - e^x) dx \quad \rightarrow \quad \frac{y^2}{2} + e^y = \frac{x^2}{2} - e^x + c_1$$

The last expression as a mixture of transcendental (i.e. exponential) and polynomial functions, so in general, it cannot be easily solved for y, therefore it is fine to report the solution in the implicit form:

$$\frac{y^2}{2} + e^y = \frac{x^2}{2} - e^x + c_1\,, \qquad \textit{for some } c_1 \in \mathbb{R}\,.$$

Question 1.8 *Write the general solution of*

$$\frac{d^2x}{dt^2} = -\omega^2 x\,, \qquad \omega \in \mathbb{R}\,.$$

Solution 1.8 *The solution can be found using the auxiliary (algebraic) equation*

$$r^2 = -\omega^2 r^0\,,$$

whose roots are $\pm i\omega$*. Then, for purely imaginary roots, the theory tells us that the general solution is given by,*

$$x(t) = A\cos(\omega t) + B\cos(\omega t)\,,$$

where A *and* B *are constants to be determined by the initial conditions.*

Question 1.9 *In triangle* ABC*,* $\overrightarrow{AB} = 6\boldsymbol{i} + 2\boldsymbol{j}$ *and* $\overrightarrow{AC} = 8\boldsymbol{i} - 5\boldsymbol{j}$*.*

a) *Find the vector* $\overrightarrow{BC}$

b) *Find the length of the line* AB

Solution 1.9 *a) To get from* A *to* C*, we can always go from* A *to* B *and* B *to* C*. If we think of* $\overrightarrow{AB}$ *as the vector from* A *to* B *and so on, this gives the natural vector equation:*

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Thus, to get $\overrightarrow{BC}$ *we rearrange:*

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$$

If we write $\overrightarrow{BA} = -\overrightarrow{AB}$ *this is the same as going from* B *to* A *and then* A *to* C*. Solving the equation gives:*

$$\overrightarrow{BC} = (8-6)\boldsymbol{i} + (-5-2)\boldsymbol{j} = 2\boldsymbol{i} - 7\boldsymbol{j}$$

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b) We use the formula for magnitude of a vector:

$$|AB| = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$

Question 1.10 *Relative to the origin* O*:*

the point A *has position vector* $2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$

the point B *has position vector* $4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

and the point C *has position vector* $a\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$*, where* $a < 0$ *is a constant.*

D *is the point such that* $\overrightarrow{AB} = \overrightarrow{BD}$*.*

a) *Find the position vector of* D

b) *Given that* $\overrightarrow{AC} = 4$*, find the value of* a

Solution 1.10 *a) First we find* $\overrightarrow{AB}$*:*

$$\begin{aligned}\overrightarrow{AB} &= (4-2)\boldsymbol{i} + (-2-3)\boldsymbol{j} + (3-(-4))\boldsymbol{k}\\ &= 2\boldsymbol{i} - 5\boldsymbol{j} + 7\boldsymbol{k}\end{aligned}$$

Let $D = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$*. We know by equating* $\overrightarrow{BD}$ *to the found* $\overrightarrow{AB}$ *that:*

$$\begin{aligned}x - 4 = 2 &\Rightarrow x = 6\\ y - (-2) = -5 &\Rightarrow y = -7\\ z - 3 = 7 &\Rightarrow z = 10\end{aligned}$$

Hence $D = (6, -7, 10)$.

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b) To find a, we first find $\overrightarrow{AC} = (a-2)\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$. Then:

$$\begin{aligned}
|\overrightarrow{AB}| = \sqrt{(a-2)^2 + 2^2 + 2^2} &= 4 \\
\Rightarrow (a-2)^2 + 8 &= 16 \\
\Rightarrow a^2 - 4a + 4 + 8 - 16 &= 0 \\
\Rightarrow a^2 - 2a - 4 &= 0 \\
\Rightarrow a &= 2 \pm 2\sqrt{2}
\end{aligned}$$

As we know $a < 0$, this means $a = 2 - 2\sqrt{2}$.

Question 1.11 *With respect to a fixed origin O, the line l_1 is given by the equation*

$$\boldsymbol{r} = \begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

where μ is a scalar parameter. The point A lies on l_1 where $\mu = 1$.

a) Find the coordinates of A.

The point P has position vector

$$\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

The line l_2 passes through the point P and is parallel to the line l_1.

b) Write down a vector equation for the line l_2.

Solution 1.11 *a) To find A we substitute $\mu = 1$ and sum the two vectors:*

$$A = \begin{pmatrix} 8-5 \\ 1+4 \\ -3+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

giving us the answer.

---

b) For l_1, we can think of $\begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix}$ as a point on the line, and $\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ dictating the slope; as we change μ we traverse the line. Thus, if l_2 is parallel it will have the same slope component (up to proportionality), but a different point; this time P. So the equation is:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

where λ is a scalar parameter like μ.

Question 1.12 *Let the following matrices be defined as*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -5 & 6 \\ 7 & 11 \\ -3 & -4 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 4 & 1 \end{bmatrix}.$$

In the following, compute the resultant value, or explain why it cannot be computed:

(a) $\boldsymbol{AB}$.

(b) $\boldsymbol{A}-\boldsymbol{D}$.

(c) $\boldsymbol{BA}$.

(d) $\boldsymbol{BC}$.

(e) $\boldsymbol{AD}+\boldsymbol{CE}$.

(f) $\boldsymbol{EB}^{\top}$.

(g) $\boldsymbol{A}^2$.

(h) $\boldsymbol{C}^2$.

(i) $|\boldsymbol{A}|$.

(j) $|\boldsymbol{B}|$.

(k) $\boldsymbol{A}^{-1}$.

(l) $\boldsymbol{D}^{-1}$.

Solution 1.12

(a) Recall that some matrix $\boldsymbol{A}$ can be post-multiplied by some matrix $\boldsymbol{B}$ (the order matters) if $\boldsymbol{A}$ is some $k\times m$ matrix and $\boldsymbol{B}$ is some $m\times n$ matrix, that is, the number of $\boldsymbol{A}$'s columns is equal to the number of $\boldsymbol{B}$'s rows, with the resultant $\boldsymbol{AB}$ being a $k\times n$ matrix; in this case, $\boldsymbol{A}$ is 2×2 and $\boldsymbol{B}$ is 3×2 and so the product $\boldsymbol{AB}$ cannot be computed as the number of $\boldsymbol{A}$'s columns (2) isn't equal to the number of $\boldsymbol{B}$'s rows (3).

(b) Recall that addition and subtraction between matrices can only occur if matrices have the same dimensions, thus the number of rows and columns of $\boldsymbol{A}$ must be equal to the number of rows and columns of $\boldsymbol{B}$, respectively; here, $\boldsymbol{A}$ and $\boldsymbol{D}$ are both 2×2 matrices, so the calculation is possible:

$$\boldsymbol{A}-\boldsymbol{D}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}-\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}=\begin{bmatrix}a_{11}-d_{11} & a_{12}-d_{12}\\ a_{21}-d_{21} & a_{22}-d_{22}\end{bmatrix}=\begin{bmatrix}1-1 & 3-3\\ 1-3 & 3-1\end{bmatrix}=\begin{bmatrix}0 & 0\\ -2 & 2\end{bmatrix}.$$

(c) The number of columns of $\boldsymbol{B}$ (2) is equal to the number of rows of $\boldsymbol{A}$ (2), so, this calculation is possible:

$$\begin{aligned}\boldsymbol{BA}&=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{bmatrix}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22}\\ b_{21}a_{11}+b_{22}a_{21} & b_{21}a_{12}+b_{22}a_{22}\\ b_{31}a_{11}+b_{32}a_{21} & b_{31}a_{12}+b_{32}a_{22}\end{bmatrix}\\ &=\begin{bmatrix}(-5)\times1+6\times1 & (-5)\times3+6\times3\\ 7\times1+11\times1 & 7\times3+11\times3\\ (-3)\times1+(-4)\times1 & (-3)\times3+(-4)\times3\end{bmatrix}=\begin{bmatrix}1 & 3\\ 18 & 54\\ -7 & -21\end{bmatrix}.\end{aligned}$$

(d) The number of columns of $\boldsymbol{B}$ (2) is equal to the number of rows of $\boldsymbol{C}$ (2, even though the rows themselves each only contain one element), so, this calculation is possible:

$$\boldsymbol{BC}=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}b_{11}c_1+b_{12}c_2\\ b_{21}c_1+b_{22}c_2\\ b_{31}c_1+b_{32}c_2\end{bmatrix}=\begin{bmatrix}(-5)\times8+6\times(-4)\\ 7\times8+11\times(-4)\\ (-3)\times8+(-4)\times(-4)\end{bmatrix}=\begin{bmatrix}-64\\ 12\\ -8\end{bmatrix}.$$

(e) Firstly, observe that $\boldsymbol{A}$ and $\boldsymbol{D}$ are 2×2 matrices, and so their product can be computed to give a resultant 2×2 matrix; secondly, observe that the number of columns of $\boldsymbol{C}$ and the number of rows of $\boldsymbol{E}$ are both 1, so their product can also be computed to be a 2×2 matrix due to the fact that the number of rows of $\boldsymbol{C}$ and the number of columns of $\boldsymbol{E}$ are both 2; thirdly, an addition of those resultant 2×2 matrices can obviously take place:

$$\begin{aligned}\boldsymbol{AD}+\boldsymbol{CE}&=\begin{bmatrix}1 & 3\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}+\begin{bmatrix}8\\ -4\end{bmatrix}\begin{bmatrix}4 & 1\end{bmatrix}=\begin{bmatrix}1\times1+3\times3 & 1\times3+3\times1\\ 1\times1+3\times3 & 1\times3+3\times1\end{bmatrix}+\begin{bmatrix}8\times4 & 8\times1\\ (-4)\times4 & (-4)\times1\end{bmatrix}\\ &=\begin{bmatrix}10 & 6\\ 10 & 6\end{bmatrix}+\begin{bmatrix}32 & 8\\ -16 & -4\end{bmatrix}=\begin{bmatrix}42 & 14\\ -6 & 2\end{bmatrix}.\end{aligned}$$

(f) Because $\boldsymbol{B}$ is a 3×2 matrix, its transpose $\boldsymbol{B}^\top$ is thus a 2×3 matrix; hence, since the number of columns of $\boldsymbol{E}$ is the same as the number of rows of $\boldsymbol{B}^\top$, the resultant product can be computed:

$$\begin{aligned}\boldsymbol{E}\boldsymbol{B}^\top &= \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{bmatrix} = \begin{bmatrix} e_1 \times b_{11} + e_2 \times b_{12} & e_1 \times b_{21} + e_2 \times b_{22} & e_1 \times b_{31} + e_2 \times b_{32} \end{bmatrix} \\ &= \begin{bmatrix} 4\times(-5) + 1\times 6 & 4\times 7 + 1\times 11 & 4\times(-3) + 1\times(-4) \end{bmatrix} = \begin{bmatrix} -14 & 39 & -16 \end{bmatrix}.\end{aligned}$$

(g) Recall that $\boldsymbol{A}^2$ is another way of expressing $\boldsymbol{A}\boldsymbol{A}$, and so for this to be suitably calculated, the number of $\boldsymbol{A}$'s rows must be equal to the number of its columns; here they are both 2 ($\boldsymbol{A}$ is square), so the calculation can be made:

$$\begin{aligned}\boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}a_{11} + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1\times 1 + 3\times 1 & 1\times 3 + 3\times 3 \\ 1\times 1 + 3\times 1 & 1\times 3 + 3\times 3 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 4 & 10 \end{bmatrix}.\end{aligned}$$

(h) Recall that, notationally, $\boldsymbol{C}^2 = \boldsymbol{C}\boldsymbol{C}$, but the number of columns of $\boldsymbol{C}$ (1) is not equal to the number of rows of $\boldsymbol{C}$ (2), so a resultant expression cannot be computed.

(i) Note that $\boldsymbol{A}$ is a 2×2 matrix, thus it is square and thus its determinant exists:

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21} = 1\times 3 - 3\times 1 = 0.$$

(j) Since $\boldsymbol{B}$ is a 3×2 matrix, it is not square, and thus it is simply not possible to compute a determinant for $\boldsymbol{B}$ as per the definition.

(k) Recall from earlier that $|\boldsymbol{A}| = 0$, which is a problem when attempting to compute the inverse for $\boldsymbol{A}$ as division by zero is not possible; thus, from the earlier observation, $\boldsymbol{A}$ is singular and has no inverse.

(l) Firstly, $\boldsymbol{D}$ is a square 2×2 matrix, so its determinant can be computed:

$$|\boldsymbol{D}| = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 1\times 1 - 3\times 3 = -8.$$

This determinant is not equal to zero, thus computing the inverse of $\boldsymbol{D}$ is viable:

$$\boldsymbol{D}^{-1} = \frac{1}{|\boldsymbol{D}|} \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix}.$$

Question 1.13 (2023 A Level Maths Paper 1 Q1) *Find the coefficient of* x^7 *in the expansion of* $(2x-3)^7$.

a) *-2187*

b) *-128*

c) *2*

d) *128*

Solution 1.13 *In general, the formula for the binomial expansion of $(a+b)^n$ is:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

If we let $n = 7$, $a = 2x$ and $b = -3$, then the x^7 term will have $k = 7$. First, we find $\binom{7}{7} = 1$. Then, we get $(2x)^7 = 128$. Finally, $b^{7-7} = 0$, and so the term for x^7 is 128.

Question 1.14 (2022 A Level Maths Paper 1 Q2) *A periodic sequence is defined by $U_n = (-1)^n$. State the period of the sequence.*

a) -1

b) -0

c) 1

d) 2

Solution 1.14 *The period of a periodic is the smallest integer ω such that $U_n = U_{n+\omega}$, when the sequence begins to repeat itself. It can also be defined as the number of terms in its repeating patter. In this case, the sequence alternates between -1 and 1 for even and odd powers of n, and so the period is 2.*

Question 1.15 (2021 A Level Maths Paper 1 Q1) *A geometric sequence has a sum to infinity of -3. A second sequence is formed by multiplying each term of the sequence by -2. What is the sum of the new sequence?*

a) The sum to infinity doesn't exist

b) -6

c) -3

d) 6

Solution 1.15 *Let T_n be the terms of the sequence such that $\sum_{n=1}^{\infty} T_n = -3$. Let $U_n = -2T_n$ be a new sequence. Then:*

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} -2T_n = -2\sum_{n=1}^{\infty} T_n = -2(-3) = 6$$

Question 1.16 (2021 A Level Maths Paper 1 Q6) *The ninth term of an arithmetic sequence is 3. The sum of the first n terms of the sequence is S_n, with $S_{21} = 42$.*

a) Find the first term and common difference of the series

A second arithmetic series has first term -18 and common difference $\frac{3}{4}$. The sum of the first n terms of the sequence is T_n.

b) Find the value of n such that $T_n = S_n$.

Solution 1.16 *a) Let a be the first term and d be the common difference. Then, using the general formula for a term we have:*

$$T_9 = 3 = a + 8d$$

We also use the formula for the sum of an arithmetic sequence to get:

$$S_{21} = 42 = \frac{21}{2}(2a + 20d)$$

This gives us two simultaneous equations:

$$3 = a + 8d \Rightarrow a = 3 - 8d$$
$$42 = 21a + 210d \Rightarrow 2 - 10d = a$$

Hence we get:

$$3 - 8d = 2 - 10d \Rightarrow 2d = -1 \Rightarrow d = -\frac{1}{2}$$
$$\Rightarrow a = 2 - 10(-\frac{1}{2}) = 7$$

---

b) It is very easy in this question (from personal experience) to mix up the notation. S_n specifically refers to the sum in a), while T_n is the sum of this second series rather than representing the general term as usual. With $a = -18$ and $d = \frac{3}{4}$ we have:

$$T_n = \frac{n}{2}[2(-18) + \frac{3(n-1)}{4}]$$

We then set $T_n = S_n$ and solve for n.

$$S_n = T_n$$
$$\frac{n}{2}[2(7) - \frac{(n-1)}{2}] = \frac{n}{2}[2(-18) + \frac{3(n-1)}{4}]$$

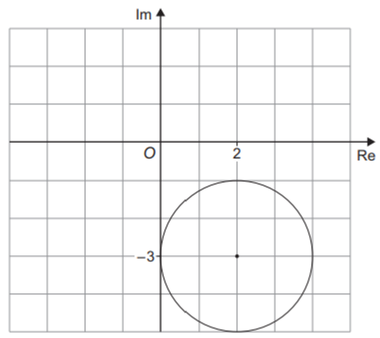
We can bring everything to one side and factorise out $\frac{n}{2}$:

$$\frac{n}{2}[14 + 36 - \frac{n-1}{4}(2+3)] = 0$$
$$\Rightarrow \frac{n}{2}[50 - \frac{5(n-1)}{4}] = 0$$

Thus $n = 0$ is a trivial solution. We then solve:

$$50 - \frac{5(n-1)}{4} = 0$$
$$\Rightarrow 40 = n - 1$$
$$\Rightarrow 41 = n$$

Question 1.17 (2023 AQA Further Maths Paper 1 Q2) *The diagram below shows a locus on an Argand diagram.*



Which of the equations below represents the locus shown above?

a) $|z - 2 + 3i| = 2$

b) $|z + 2 - 3i| = 2$

c) $|z - 2 + 3i| = 4$

d) $|z + 2 - 3i| = 4$

Solution 1.17 *A locus is the path represented by some algebraic expression of a complex number. In this case, we have something of the form:*

$$|z - (a + bi)| = r$$

which gives a circle; a set of complex numbers that are a fixed distance/radius r from some complex number $a + bi$, the centre of the circle. Thus, there are two steps to deciphering the solution:

i) Find the centre of the circle $a + bi$

ii) Find the radius of the circle r.

By inspecting the diagram, the centre of the circle is clearly the point $z = 2 - 3i$ and thus the locus is of the form:

$$|z - (2 - 3i)| = r$$
$$\Rightarrow |z - 2 + 3i| = r$$

We then note the point $z = 0 - 3i$ appears to be on the locus, and so the radius $r2$. Thus, the locus is of the form $|z - 2 + 3i| = 2$.

Question 1.18 (2023 AQA Further Maths Paper 1 Q6(a)) *Express $-5 - 5i$ in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$*

Solution 1.18 *First, we find the radius r using Pythagoras' Theorem:*

$$r = \sqrt{5^2 + 5^2}$$
$$r = \sqrt{50} = 5\sqrt{2}$$

Then, let ϕ be the angle such that:

$$\tan\phi = \frac{5}{5} = 1$$

and thus:

$$\phi = \tan^{-1}(1) = \frac{\pi}{4}$$

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If our complex number was $= 5 + 5i$, then $\theta = \phi$ and we're done. However, we need to account for which quadrant the complex number lies in. With negative real and imaginary components, we have that $\theta = -\pi + \phi = -\frac{3\pi}{4}$ and thus:

$$z = 5\sqrt{2}e^{-\frac{3\pi i}{4}}$$

2 Further

Question 2.1 (2019 AQA A-Level Paper 1 Q6) *The function f is defined by*

$$f(x) = \tfrac{1}{2}(x^2 + 1), x \geq 0$$

a) Find the range of f

b) Find $f^{-1}(x)$

c) State the range of $f^{-1}(x)$

d) State the transformation which maps the graph $y = f(x)$ onto the graph $y = f^{-1}(x)$

e) Find the coordinates of the point of intersection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Solution 2.1 *(a) We look at both endpoints of the domain. First, $f(0) = \frac{1}{2}$. As for the other endpoint, it is ∞, and so the endpoint of the range is also ∞. hence, the range is $[\frac{1}{2}, \infty]$*

(b) We solve for x in:

$$y = \tfrac{1}{2}(x^2 + 1) \Rightarrow 2y = x^2 + 1$$
$$2y - 1 = x^2 \Rightarrow x = \sqrt{2y - 1}$$

Thus $f^{-1}(x) = \sqrt{2x - 1}$ for $x \geq \frac{1}{2}$.

(c) The domain of f^{-1} should be the range of f. Similarly, the range of f^{-1} should be the domain of f, $[0, \infty]$.

(d) In order to transform a function into its inverse, you reflect across the line $y = x$.

(e) To find the coordinates of the intersection, we can let them both equal to each other:

$$\tfrac{1}{2}(x^2 + 1) = \sqrt{2x - 1}$$

However, its easier to use the answer to part (d): because we reflect along $y = x$, if they intersect, it must be on this line. Thus we solve:

$$\begin{aligned} x &= \tfrac{1}{2}(x^2+1) \\ \Rightarrow x^2 - 2x + 1 &= 0 \\ \Rightarrow (x-1)(x-1) = 0 \Rightarrow x &= 1 \end{aligned}$$

There is only one point of intersection when $x = 1$ and thus $y = 1$. Hence, the point is $(1, 1)$.

Question 2.2 (2022 TMUA Paper 1 Q1) *How many solutions are there to the equation*

$$2\cos^4(\theta) - 5\cos^2(\theta) + 3 = 0$$

Solution 2.2 *Let $x = \cos^2(\theta)$. The equation then becomes:*

$$\begin{aligned} &2x^2 - 3x + 3 = 0 \\ \Rightarrow &(2x-3)(x-1) = 0 \end{aligned}$$

Hence, we have that $\cos^2(\theta) = \frac{3}{2}$ or $\cos^2(\theta) = 1$. The first equation has no real solutions, and so we have only the solutions of $\cos(\theta) = \pm 1$. The equation $\cos(\theta) = 1$ has solutions at $\theta = 0$ and $\theta = 2\pi$. The equation $\cos(\theta) = -1$ gives $\theta = \pi$. Hence, there are 3 solutions.

Question 2.3 (2022 TMUA Paper 2 Q5) *A straight line L passes through $(1, 2)$. Let P be the statement: "**if** the y-intercept of L is negative, **then** the x-intercept of L is positive."*

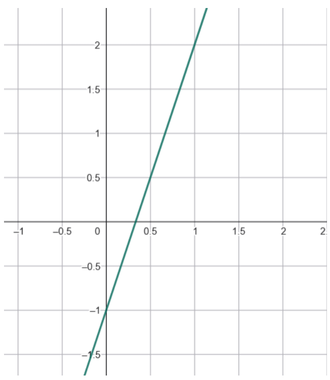
*Which of the following statements **must** be true? (note: multiple can be true)*

a) P

b) The converse of P

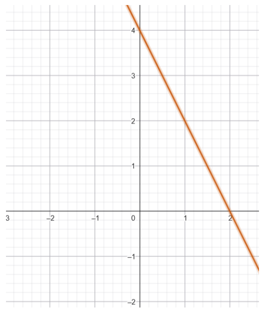
c) The contrapositive of P

Solution 2.3 *It is useful to draw a possible diagram here (e.g y = 2x - 1) to help your understanding, even if it is not the actual line L.*



The example suggests P is true, but we can also prove it. If $y < 0$ when $x = 0$ and $y > 0$ when $x = 1$, there must be some point $0 < x < 1$ where $y = 0$.

*Next we check the converse of P, which states: "**if** the x-intercept of L is positive, **then** the y-intercept of L is negative." Well, we prove this is wrong by way of a counterexample:*



In fact, if the x-intercept is greater than 1, then this will not be true.

Finally, the contrapositive of P has the same truth value, and so if P is true so is the contrapositive.

Question 2.4 (2018 Maths Paper 3 Q2) *A curve has equation $y = x^5 + 4x^3 + 7x + q$, where q is a positive constant. Find the gradient of the curve at the point $x = 0$.*

Solution 2.4 *To find the gradient we find $\frac{\mathrm{d}y}{\mathrm{d}x}$. This is:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4 + 12x^2 + 7$$

When $x = 0$, the gradient is thus 7.

Question 2.5 (2024 Maths Paper 3 Q4) *A curve has equation $y = x^4 + 2^x$. Find an expression for $\frac{dy}{dx}$.*

Solution 2.5 *The derivative of x^4 is $4x^3$. We also have the identity:*

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \log(2)$$

and so:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 + 2^x \log(2)$$

Question 2.6 (2018 Maths Paper 1 Q5) *A curve is defined by the parametric equations*

$$x = \frac{4}{2^t} + 3$$
$$y = 3 \times 2^t - 5$$

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4} \times 2^{2t}$

Solution 2.6 *We first find the derivative of both x and y with respect t. Start with y:*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3 \times 2^t \times \log(2)$$

For x, we can use the quotient rule. However, the chain rule with $u = 2^t$ and $x = \frac{4}{u} + 3$ is faster. This gives:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{4}{(2^t)^2} \times 2^t \times \log(2) = -\frac{4}{2^t} \times \log(2)$$

Then, $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be the ratio of these two derivatives:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{3 \times 2^t \times \cancel{\log(2)}}{\frac{4}{2^t} \times \cancel{\log(2)}} \\ &= -\frac{3}{4} \times 2^{t+t} = -\frac{3}{4} \times 2^{2t}\end{aligned}$$

Question 2.7 *Solve $\int e^x \cos(x) \mathrm{d}x$*

Solution 2.7 *This integral is a classic case for integration by parts twice, as covered in the refresher class/sheet. Set*

$$I = \int e^x \cos(x)\,\mathrm{d}x.$$

Start with

$$\begin{array}{l|l} u = e^x & \mathrm{d}v = \cos(x)\,\mathrm{d}x \\ \mathrm{d}u = e^x\,\mathrm{d}x & v = \sin(x) \end{array}$$

Then

$$I = e^x \sin(x) - \int e^x \sin(x)\,\mathrm{d}x.$$

Denote the remaining integral by $J = \int e^x \sin(x)\,\mathrm{d}x$ *and integrate it by parts again:*

$$\begin{array}{l|l} u = e^x & \mathrm{d}v = \sin(x)\,\mathrm{d}x \\ \mathrm{d}u = e^x\,\mathrm{d}x & v = -\cos(x) \end{array}$$

Hence

$$J = -e^x \cos(x) + \int e^x \cos(x)\,\mathrm{d}x \;=\; -e^x\cos(x) + I.$$

Substitute this back into the expression for I*:*

$$I = e^x \sin(x) - J \;=\; e^x\sin(x) - \big(-e^x\cos(x) + I\big) = e^x\sin(x) + e^x\cos(x) - I.$$

Therefore

$$2I = e^x\big(\sin x + \cos x\big) \quad \Rightarrow \quad I = \int e^x\cos(x)\,\mathrm{d}x = \frac{e^x}{2}\big(\sin x + \cos x\big) + C$$

Question 2.8 (2024 AQA Maths, Paper 1 Q18) *Use a suitable substitution to show that* $\int_0^4 (4x+1)\sqrt{2x+1}\mathrm{d}x$ *can be written as* $\frac{1}{2}\int_a^9 (2u^{\frac{3}{2}} - u^{\frac{1}{2}})\mathrm{d}u$*, where* a *is a constant to be found.*

Hence, or otherwise, show that:

$$\int_0^4 (4x+1)\sqrt{2x+1}\mathrm{d}x = \frac{1322}{15}$$

Solution 2.8 *It is natural to choose to substitute something inside another function, so target the inside of the square root: let*

$$u = 2x+1, \quad x = \tfrac{u-1}{2} \qquad \Longleftrightarrow \qquad \mathrm{d}u = 2\,\mathrm{d}x$$

The limits change as $2(4)+1 = 9$ *(given to us) and* $2(0)+1 = 1 = a$*. Moreover,*

$$4x+1 = 4\cdot\frac{u-1}{2} + 1 = 2u-1, \qquad \sqrt{2x+1} = \sqrt{u} = u^{1/2}.$$

Hence

$$\int_0^4 (4x+1)\sqrt{2x+1}\,\mathrm{d}x = \int_1^9 (2u-1)\,u^{1/2}\,\frac{1}{2}\,\mathrm{d}u = \frac{1}{2}\int_1^9 \big(2u^{3/2} - u^{1/2}\big)\,\mathrm{d}u,$$

which is of the required form with $a = 1$*.*

For the value,

$$\begin{aligned} \frac{1}{2}\int_1^9 \big(2u^{3/2} - u^{1/2}\big)\,\mathrm{d}u &= \frac{1}{2}\left[2\cdot\frac{u^{5/2}}{5/2} \;-\; \frac{u^{3/2}}{3/2}\right]_1^9 \\ &= \frac{1}{2}\left[\frac{4}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9 \end{aligned} \tag{1}$$

Using $9^{3/2} = 27$ *and* $9^{5/2} = 243$*,*

$$\frac{1}{2}\left[\frac{4}{5}(243-1) - \frac{2}{3}(27-1)\right] = \frac{1}{2}\left(\frac{968}{5} - \frac{52}{3}\right) = \frac{1}{2}\cdot\frac{2644}{15} = \frac{1322}{15}$$

Question 2.9 *Naively, "evaluate"* $\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x$ *by finding the indefinite integral and plugging in the bounds. Draw a graph to convince yourself that this answer is in fact wrong, and use improper integration to find the correct answer.*

Solution 2.9 *Naive attempt (why it fails). Treating this as an ordinary definite integral and plugging in an anti-derivative gives*

$$\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x \overset{\text{naively}}{=} \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

This is already suspicious: the integrand is non–negative on $[-1, 1]$, *so the integral cannot be negative. The mistake is that the integrand is* not defined *at* $x = 0$ *(vertical asymptote), so this is an improper integral; you cannot apply the anti-derivative across the singularity.*

Graphical check. A sketch of $y = \frac{1}{x^2}$ *shows a vertical asymptote at* $x = 0$ *and symmetry about the* y*-axis; the area near* $x = 0$ *grows without bound, suggesting potential divergence.*

Proper treatment (improper integral). Split at the singularity and use limits:

$$\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\left(\int_{-1}^{-\varepsilon} \frac{1}{x^2}\,\mathrm{d}x \;+\; \int_{\varepsilon}^{1} \frac{1}{x^2}\,\mathrm{d}x\right).$$

Compute each piece:

$$\int_{\varepsilon}^{1} x^{-2}\,\mathrm{d}x = \left[-x^{-1}\right]_{\varepsilon}^{1} = -1 + \frac{1}{\varepsilon},$$

$$\int_{-1}^{-\varepsilon} x^{-2}\,\mathrm{d}x = \left[-x^{-1}\right]_{-1}^{-\varepsilon} = \frac{1}{\varepsilon} - 1.$$

Add and take the limit:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x &= \lim_{\varepsilon\to 0^+}\left(\frac{1}{\varepsilon} - 1 \;+\; \frac{1}{\varepsilon} - 1\right)\\ &= \lim_{\varepsilon\to 0^+}\left(\frac{2}{\varepsilon} - 2\right) = +\infty.\end{aligned}$$

Thus the integral diverges *(it does not exist as a finite improper integral).*

Question 2.10 (AQA Specimen Paper 1, Q8b) *Find the exact value of* $\int_0^1 2^x\sqrt{3+2^x}\mathrm{d}x$. *Fully justify your answer.*
Hint: use the previous question and a substitution.

Solution 2.10 *Use the previous result* $\int a^x\,\mathrm{d}x = \dfrac{a^x}{\log a} + C$ *and the substitution*

$$u = 3 + 2^x \quad\Longleftrightarrow\quad \mathrm{d}u = (\log 2)\,2^x\,\mathrm{d}x \quad\Longleftrightarrow\quad 2^x\,\mathrm{d}x = \frac{1}{\log 2}\,\mathrm{d}u$$

When $x = 0$, $u = 3 + 2^0 = 4$; *when* $x = 1$, $u = 3 + 2^1 = 5$. *Hence*

$$\begin{aligned}\int_0^1 2^x\sqrt{3+2^x}\,\mathrm{d}x &= \int_{u=4}^{5} \sqrt{u}\,\frac{1}{\log 2}\,\mathrm{d}u\\ &= \frac{1}{\log 2}\int_4^5 u^{1/2}\,\mathrm{d}u\\ &= \frac{1}{\log 2}\left[\frac{2}{3}u^{3/2}\right]_4^5\\ &= \frac{2}{3\log 2}\left(5^{3/2} - 4^{3/2}\right).\end{aligned}$$

Since $5^{3/2} = 5\sqrt{5}$ *and* $4^{3/2} = 8$, *the exact value is*

$$\int_0^1 2^x\sqrt{3+2^x}\,\mathrm{d}x = \frac{2}{3\log 2}\left(5\sqrt{5} - 8\right).$$

Question 2.11 (June 2024 — Maths Paper 1 Q20)
A gardener stores rainwater in a cylindrical container. The container has a height of 130 centimetres. The gardener empties the water from the container through a hose. The hose is attached 5 centimetres from the bottom of the container. At time t *minutes after the hose is switched on, the depth of water,* h *centimetres, in the container decreases at a rate which is proportional to* $h - 5$. *Initially the container of water is full, and the depth of water is decreasing at a rate of 1.5 centimetres per minute*

(a) Show that

$$\frac{dh}{dt} = -0.012(h-5)$$

(b) Solve the equation to find an expression of h in terms of t.

(c) Find the time taken for the container to be half empty. Give your answer to the nearest minute.

Solution 2.11

(a) The problem states that the change on the depth of water, h, is proportional to $h-5$, which implies that

$$\frac{dh}{dt} = -\gamma(h-5)\,, \qquad \gamma \in \mathbb{R}^+\,.$$

Note: The minus sign takes into account that h decreases.
Since the initial decreasing rate of h is $1.5\,cm\,min^{-1}$ (i.e. $\dot{h}(t_0) = -1.5$)[1] and the initial value of h is 130 cm (i.e. $h(t_0) = 130$), it implies that,

$$-1.5 = -\gamma(130-5) \qquad \longrightarrow \gamma = \frac{1.5}{125} = 0.012$$

(b) The differential equation can be solved by direct integration,

$$\frac{dh}{dt} = -\gamma(h-5) \quad \to \quad \frac{1}{h-5}\frac{dh}{dt} = -\gamma \quad \to \quad \int \frac{1}{h-5}\frac{dh}{dt}dt = -\gamma \int dt \quad \to \quad \int \frac{dh}{h-5} = -\gamma \int dt\,,$$

where in the last step we made a change of variable on the left hand integral using dh/dt. There follows,

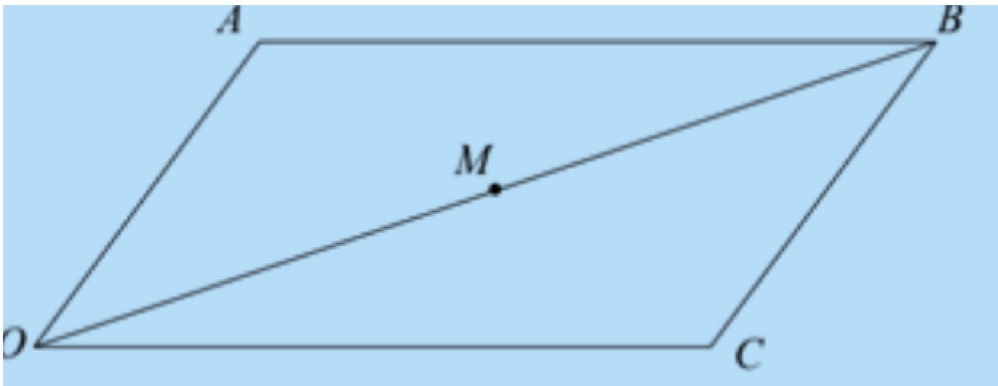
$$\int_{130}^{h} \frac{d\tilde{h}}{\tilde{h}-5} = -\gamma \int_{t_0}^{t} d\tilde{t} \quad \to \quad \ln\left(\frac{h-5}{130-5}\right) = -0.012(t-t_0) \quad \to \quad h(t) = 125\,e^{-0.012(t-t_0)}$$

(c) If $h(t_1) = 130/2 = 65$ and we set $t_0 = 0$, there follows that,

$$\ln\left(\frac{65-5}{130-5}\right) = -0.012\,t_1 \quad \to \quad t_1 = \frac{1}{0.012}\ln\left(\frac{125}{60}\right) \approx 61.164$$

Therefore the cylinder will be half empty around 61 min.

Question 2.12 *$OABC$ is a parallelogram with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \mathbf{c}$. M is the midpoint of $\overrightarrow{OB}$.*



*a) Find, in terms of **a** and **c**, simplifying your answers:*

[1]**Notation:** $\dot{h} = \frac{dh}{dt}$.

i) $\overrightarrow{AC}$

ii) $\overrightarrow{OM}$

b) Hence prove the diagonals of a parallelogram bisect one another

Solution 2.12 *(a) (i) Use the formula* $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ *to get* $\overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{AC}$ *and so* $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$.

(ii) We know that $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OC} + \overrightarrow{OA} = \boldsymbol{c} + \boldsymbol{a}$. *Then,* $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{a}$.

(b) We should try to use both of the answers we found previously, if possible. We do some algebra with the vectors, noting that reversing vectors changes signs:

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OM}\\ &= -\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{c} + \tfrac{1}{2}\boldsymbol{a}\\ &= \tfrac{1}{2}\boldsymbol{c} - \tfrac{1}{2}\boldsymbol{a} = \tfrac{1}{2}\overrightarrow{AC}\end{aligned}$$

as required.

Question 2.13 *With respect to a fixed origin O, the lines l_1 and l_2 are given by:*

$$l_1 : \boldsymbol{r} = \begin{pmatrix}4\\28\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\-5\\1\end{pmatrix}$$

and

$$l_2 : \boldsymbol{r} = \begin{pmatrix}5\\3\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-4\end{pmatrix}$$

where λ and μ are scalar parameters. The lines intersect at the point X.

(a) Find the coordinates of X

(b) Find the size of the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places.

The point A lies on l_1 and has position vector:

$$\begin{pmatrix}2\\18\\6\end{pmatrix}$$

(c) Find the distance AX, giving your answer as a surd in its simplest form.

The point Y lies on l_2. Given that the vector $\overrightarrow{YA}$ is perpendicular to l_1:

(d) find the distance of YA, giving your answer to one decimal place.

The point B lies on l_1 where $|\overrightarrow{AX}| = 2|\overrightarrow{AB}|$.

(e) Find the two possible position vectors of B.

Solution 2.13 *(a) The two lines intersect when they are equal. Generally, as there are two unknowns (λ and μ) we would usually have to equate two elements and create a simultaenous equation. However, the second element of l_2's slope movement vector is 0, and so we can simply solve:*

$$28 - 5\lambda = 3 \Rightarrow \lambda = 5$$

and so the point X is:

$$\begin{pmatrix}4-(5)\\28-5(5)\\4+(5)\end{pmatrix} = \begin{pmatrix}-1\\3\\9\end{pmatrix}$$

Solving for μ would also work.

(b) To find the angle between the lines, we represent them as vectors and find the dot product between the two. We then rearrange the formula for the dot product. First, we represent them each by their vector component and find the magnitudes:

$$\boldsymbol{d}_1 = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \Rightarrow |\boldsymbol{d}_1| = \sqrt{(-1)^2 + (-5)^2 + 1^2} = \sqrt{27}$$
$$\boldsymbol{d}_2 = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \Rightarrow |\boldsymbol{d}_2| = \sqrt{(3)^2 + (0)^2 + (-4)^2} = 5$$

We now find the dot product of the two:

$$\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = (-1)(3) + (-5)(0) + (1)(-4) = -7$$

Hence, the formula gives:

$$\begin{aligned} \boldsymbol{d}_1 \cdot \boldsymbol{d}_2 &= |\boldsymbol{d}_1||\boldsymbol{d}_2| \cos(\theta) \\ \Rightarrow \cos(\theta) &= \frac{\boldsymbol{d}_1 \cdot \boldsymbol{d}_2}{|\boldsymbol{d}_1||\boldsymbol{d}_2|} \\ \Rightarrow \cos\theta &= \frac{-7}{\sqrt{27}} \\ \Rightarrow \theta &= \cos^{-1}(-0.2694) = 105.63 \end{aligned}$$

However, we were asked for the acute angle and so we get $180 - 105.63 = 74.37^o$.

(c) As we've seen before, we manipulate to get $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$. *The vector* $\overrightarrow{OA}$ *is just the point* A *and similar for* $\overrightarrow{OX}$*, so:*

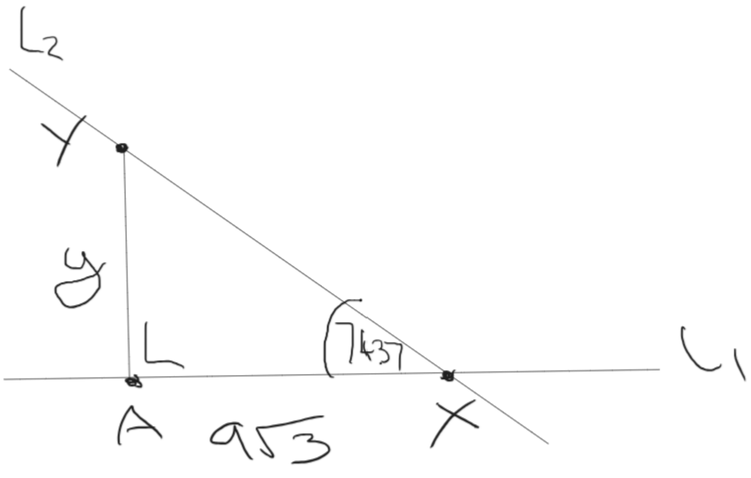
$$\begin{aligned} \overrightarrow{AX} &= \overrightarrow{OX} - \overrightarrow{OA} \\ &= \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix} \end{aligned}$$

We then find the magnitude:

$$|AX| = \sqrt{(-3)^2 + (15)^2 + 3^2} = \sqrt{243} = 9\sqrt{3}$$

where we know to simplify further because of the question asking. We could have also solved this using the distance between two points formula which is just a compact version of our approach.

(d)Here we can use the fact the line is perpendicular to A *to draw a right angled triangle, where* l_1 *and* l_2 *make up the other sides:*



Thus, we have the opposite and adjacent for the known angle, so we use tan:

$$\tan(74.37) = \frac{y}{9\sqrt{3}} \Rightarrow y = 55.7$$

Done.

(e) We know that B lies on l_1 so we want to find the possible values of λ that would satisfy the given condition. Furthermore, we know that A also lies on the line l_1 and has $\lambda = 2$ (you can solve any of the three components). We already found that X is on the line with $\lambda = 5$.
One possibility for B is that is is the midpoint between A and X; in this case, we would have $\lambda = 3.5$. Subbbing $\lambda = 3.5$ into the formula for l_1 gives:

$$\overrightarrow{OB} = \begin{pmatrix} 4 - 3.5 \\ 28 - 3.5(5) \\ 4 + 3.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix}$$

The other option is that B is the same distance away from A but in the opposite direction to X. We increased $\lambda = 2$ by 1.5 to get to B in the first case, so in the second we subtract 1.5 to get $\lambda = 0.5$.

$$\overrightarrow{OB} = \begin{pmatrix} 4 - 0.5 \\ 28 - 0.5(5) \\ 4 + 0.5 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}$$

Question 2.14 (2023 Further Maths Paper 2 Q8) *Let $\boldsymbol{A}$ be a non-singular 2×2 matrix with $\boldsymbol{A}^\top$ being the transpose of $\boldsymbol{A}$.*

(a) *Using the result*

$$(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top,$$

show that

$$(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}.$$

(b) *It is given that $\boldsymbol{A} = \begin{bmatrix} 4 & 5 \\ 1 & k \end{bmatrix}$, where k is a real constant.*

(b) (i) *Find $(\boldsymbol{A}^{-1})^\top$ in terms of k.*

(b) (ii) *What is the restriction on possible values of k?*

Solution 2.14

(a) Define $\boldsymbol{C} = (\boldsymbol{A}^{-1})^\top$ and firstly consider post-multiplying this by $\boldsymbol{A}^\top$, which becomes

$$\boldsymbol{C}\boldsymbol{A}^\top = (\boldsymbol{A}^{-1})^\top \boldsymbol{A}^\top = (\boldsymbol{A}\boldsymbol{A}^{-1})^\top = \boldsymbol{I}^\top = \boldsymbol{I},$$

where the second equality has arisen through the original result given ($\boldsymbol{A}^{-1}$ in place of B) and the fact that the identity matrix is symmetric; secondly, considering pre-multiplying $\boldsymbol{C}$ by $\boldsymbol{A}^\top$ gives

$$\boldsymbol{A}^\top\boldsymbol{C} = \boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^{-1}\boldsymbol{A})^\top = \boldsymbol{I}^\top = \boldsymbol{I},$$

using analogous logic; hence, because $\boldsymbol{C}\boldsymbol{A}^\top = \boldsymbol{A}^\top\boldsymbol{C} = \boldsymbol{I}$, the matrix $\boldsymbol{C}$ satisfies the definition of the inverse of $\boldsymbol{A}^\top$, which is the same as saying that $\boldsymbol{C} = (\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}$.

(b) (i) With a view to computing the value of the expression $(\boldsymbol{A}^{-1})^\top$ directly by computing $\boldsymbol{A}^{-1}$ then transposing it, the first step is to determine $\boldsymbol{A}$'s determinant:

$$|\boldsymbol{A}| = \begin{vmatrix} 4 & 5 \\ 1 & k \end{vmatrix} = 4\times k - 5\times 1 = 4k - 5.$$

The second step is to then compute the inverse of $\boldsymbol{A}$, inserting the above expression:

$$\boldsymbol{A}^{-1} = \frac{1}{4k-5}\begin{bmatrix} k & -5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{k}{4k-5} & -\frac{5}{4k-5} \\ -\frac{1}{4k-5} & \frac{4}{4k-5} \end{bmatrix}.$$

The final step is to transpose the resultant matrix:

$$(\boldsymbol{A}^{-1})^\top = \begin{bmatrix} \frac{k}{4k-5} & -\frac{5}{4k-5} \\ -\frac{1}{4k-5} & \frac{4}{4k-5} \end{bmatrix}^\top = \begin{bmatrix} \frac{k}{4k-5} & -\frac{1}{4k-5} \\ -\frac{5}{4k-5} & \frac{4}{4k-5} \end{bmatrix} = \frac{1}{4k-5}\begin{bmatrix} k & -1 \\ -5 & 4 \end{bmatrix}.$$

(b) (ii) The matrix $(\boldsymbol{A}^{-1})^\top$ can only be suitably defined if the denominator $\frac{1}{4k-5}$ can be defined, which means that

$$4k - 5 \neq 0 \quad \iff \quad 4k \neq 5 \quad \iff \quad k \neq \frac{5}{4}.$$

Another way of thinking about this is to realise that $\boldsymbol{A}^\top$ must not be singular, because using the result from earlier, $(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}$; so it must hold that $|\boldsymbol{A}^\top| \neq 0$, but since it can be easily seen that through the definition of the determinant that $|\boldsymbol{A}^\top| = |\boldsymbol{A}| = 4k - 5$, then by the same rearrangement as above, $k \neq \frac{5}{4}$.

Question 2.15 (2023 A Level Maths Paper 1 Q11) *The n^{th} term of a sequences is u_n, with the sequence defined by*

$$u_{n+1} = pu_n + 70$$

where $u_1 = 400$ and p is constant.

a) *Find an expression in terms of p for u_2*

b) *It is given that $u_3 = 382$. Show that p satisfies the equation*

$$200p^2 + 35p - 156 = 0$$

c) *Given that the sequence is a decreasing sequence, find the value of u_4 and the value of u_5*

d) *The limit of u_n as n tends towards infinity is L. Write down an equation for L*

e) *Find the value of L*

Solution 2.15 *a) We use the formula to get $u_2 = 400p + 70$.*

---

b) We use the formula again to get

$$\begin{aligned} u_3 = 382 &= pu_2 + 70 = p(400p + 70) + 70 \\ \Rightarrow 382 &= 400p^2 + 70p + 70 \\ \Rightarrow 0 &= 400p^2 + 70p - 312 \\ \Rightarrow 0 &= 200p^2 + 35p - 156 \end{aligned}$$

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c) First, we solve the above quadratic equation for the two possible values of p using the quadratic formula. The values for p are $p = 0.8$ and $p = -0.975$. If we take $p = -0.975$, the series will not be decreasing, which we can check:

$$u_4 = -0.975(382) + 70 = -372.45 + 70 = -302.45$$
$$u_5 = -0.975(-302.45) + 70 = 364.89$$

Thus, we use $p = 0.8$ and get $u_4 = 375.6$ and $u_5 = 370.48$.

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d) Let $n \to \infty$, then $u_{n+1} = u_n = L$ and so $L = pL + 70$.

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e) Let $p = 0.8$ and rearrange the above equation to get $L = \frac{70}{0.2} = 350$.

Question 2.16 (2022 A Level Maths Paper 1 Q9) *The first three terms of an arithmetic sequence are given by:*

$$2x + 5 \qquad 5x + 1 \qquad 6x + 7$$

- *a) Show that $x = 5$ is the only value which gives an arithmetic sequence*
- *b) Write down the value of the first term of the sequence*
- *c) Find the common difference of the sequence*
- *d) Find N such that the sum of the first N terms of the arithmetic sequence is S_N and*

$$S_N < 100,000$$
$$S_{N+1} > 100,000$$

Solution 2.16 *a) To be an arithmetic sequence we need an equal common difference. This means:*

$$5x + 1 - (2x + 5) = 6x + 7 - (5x + 1)$$
$$3x - 4 = x + 6$$
$$2x = 10$$
$$x = 5$$

---

b) $u_1 = 2(5) + 5 = 15$

---

c) The common difference is $3x - 4 = 11$.

---

d) The formula for S_N is:

$$S_N = \frac{N}{2}(2(15) + (11 - 1)N)$$
$$= \frac{N}{2}(30 + 10N) = 5N(3 + N)$$

We now solve this for equal to 100,00; it should not give an integer, but rounding down will give us N.

$$5N(3 + N) = 100,000 \Rightarrow 3N + N^2 = 20,000$$
$$\Rightarrow N^2 + 3N - 20,000 = 0$$
$$\Rightarrow N = 139.93 \text{ or } N = -142.93$$

and so $N = 139.93$.

Question 2.17 (2022 A Level Maths Paper 1 Q12) *A geometric sequence has first term 1 and common ratio $\frac{1}{2}$.*

a) *Find the sum to infinity of the sequence*

b) *Hence, or otherwise, evaluate*

$$\sum_{n=}^{\infty} (\sin 30^o)^n$$

Solution 2.17 *(a) For a geometric sequence with first term a and common ratio $-1 < r < 1$, the inifnite geometric sum is:*

$$S = \frac{a}{1-r}$$

and thus in this case we have the infinite sum is 2.

(b) We know that $\sin 30^o = \frac{1}{2}$, and so this series is just another expression of a geometric series with common ratio $\frac{1}{2}$ but first term $\frac{1}{2}$. Thus, the total sum is 1 *rather than* 2.

Question 2.18 (2021 A Level Further Maths Paper 1 Q1) *Find*

$$\sum_{r=1}^{20} r^2 - 2r$$

from the below answers:

a) *2,450*

b) *2,660*

c) *5,320*

d) *43,680*

Solution 2.18 *We know that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Thus:*

$$\sum_{r=1}^{20}(-2r) = -2\sum_{r=1}^{20} r = -(20)(21) = -420$$

We also know that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and so:

$$\sum_{r=1}^{20} r^2 = \frac{20(21)(41)}{6} = 2870$$

and so the total sum is $2870 - 420 = 2450$.

Question 2.19 (AQA 2019 FM Paper 1 Q4) *Solve the equation* $2z - 5iz^* = 12$

Solution 2.19 *This question tests a lot of knowledge about complex numbers. First, that z^* or $\overline{z}$ is the* ***conjugate*** *such that if $z = a + bi$ then $z^* = a - bi$. The conjugate is the reflection across the real axis. Thus we can rewrite the equation (using the property $i^2 = -1$) as:*

$$\begin{aligned} 2(a+bi) - 5i(a-bi) &= 12 \\ \Rightarrow 2a + 2bi - 5ai + 5bi^2 - 12 &= 0 \\ \Rightarrow 2a - 5b - 12 + (2b - 5a) &= 0 \end{aligned}$$

We then separate the real and imaginary parts to get two simulatenous equations:

$$\begin{aligned} 2a - 5b - 12 &= 0 \\ 2b - 5a &= 0 \end{aligned}$$

This gives the final solution $z = -\frac{8}{7} - \frac{20}{7}i$.

Question 2.20 (AQA 2019 FM Paper 1 Q8) *a) If $z = cos\theta + isin\theta$, use de Moivre's theorem to prove that*

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

b) Express $\sin^5\theta$ in terms of $\sin 5\theta, \sin 3\theta$ and $\sin\theta$
c) Hence show that:

$$\int_0^{\frac{\pi}{3}} \sin^5(\theta)d\theta = \frac{53}{480}$$

Solution 2.20 *a) The polar form of a complex number is written $z = r[cos\theta + i\sin\theta]$. DeMoivre's Theorem is a quick way of taking powers of complex numbers, stating:*

$$z^n = (r[cos\theta + i\sin\theta])^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

The key to this question is noting that $\frac{1}{z^n} = z^{-n}$ and so:

$$z^n - z^{-n} = r^n[\cos(n\theta) + i\sin(n\theta)] - r^{-n}[\cos(-n\theta) + i\sin(-n\theta)]$$

However, we have that $z = \cos\theta + i\sin\theta$ and so $r = 1$. Thus:

$$z^n - z^{-n} = \cos(n\theta) + i\sin(n\theta) - \cos(-n\theta) - i\sin(-n\theta)$$

We then use the fact that $\cos$ is an even function (so $\cos(-\theta) = \cos(\theta)$) and $\sin$ is an odd function ($\sin(-\theta) = -\sin(\theta)$) to get:

$$\begin{aligned} z^n - z^{-n} &= \cos(n\theta) - \cos(n\theta) + i[\sin(n\theta) + \sin(n\theta)] \\ &= 2i\sin(n\theta) \end{aligned}$$

as required.

---

b) This question is difficult to begin without the first part, which gives a clue how to proceed. We know that $z - \frac{1}{z} = 2i\sin(\theta)$ and thus we want to take this to power 5. We then use the Binomial expansion formula:

$$(z - z^{-1})^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$$

Note that we can then group the RHS into three different expressions of the form $z^n - z^-n$, for $n = 1, 3$ and 5.

$$(z - z^{-1})^5 = (z^5 - z^-5) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

This allows us to use (a) to get:

$$\begin{aligned} (2i\sin(\theta))^5 &= 2i\sin(5\theta) - 5(2i\sin(3\theta)) + 10(2i\sin(\theta)) \\ 2^5 i^5 \sin^5(\theta) &= 2i[\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)] \end{aligned}$$

We can write $i^5 = i^2 i^2 i = (-1)(-1)i = i$ and thus cancel the i on both sides to get, after dividing by 2^5:

$$\sin^5(\theta) = \frac{1}{16}[\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)]$$

---

c) The beauty of (b) is that we have rewritten an expression which is difficult to integrate into a sum of functions which are easily integrable. In particular, the antiderivative of $\sin(n\theta) = -\frac{1}{n}cos(n\theta)$ and so:

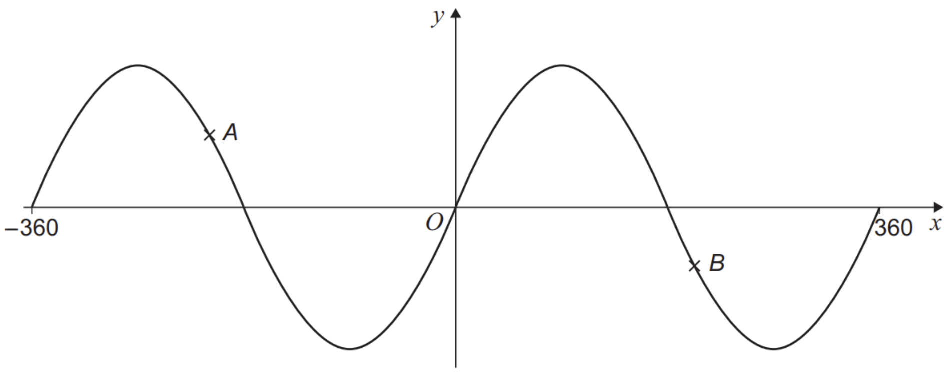
$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sin^5(\theta)d\theta &= \int_0^{\frac{\pi}{3}} \frac{1}{16}[\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)]d\theta \\ &= \left[-\frac{1}{80}\cos(5\theta) + \frac{5}{48}\cos(3\theta) - \frac{5}{8}\cos(\theta)\right]_0^{\frac{\pi}{3}} \\ &= \frac{53}{480} \end{aligned}$$

3 Advanced

Question 3.1 (2023 AQA A-Level Paper 1 Q10) *The curve with equation*

$$y = \sin(x)^o$$

for $-360 \leq x \leq 360$ *is shown below*



Point A *on the curve has coordinates* $(a, 0.5)$.

a) *Find the value of* a

b) *State the value of* $\sin(180^o - a^o)$

Point B *on the curve has coordinates* $(b, -\frac{3}{7})$.

c) *Find the exact value of* $\sin(b^o - 180^o)$

d) *Find the exact value of* $\cos(b^o)$

Solution 3.1 *Click* here *for video.*

Question 3.2 (2023 AQA Further Maths A-Level Paper 1 Q7) *The function* f *is defined by:*

$$f(x) = |\sin(x) + \tfrac{1}{2}|, 0 \leq x \leq 2\pi$$

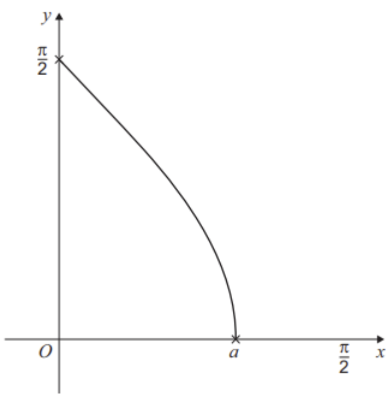
Find the set of values of x *for which* $f(x) \geq \frac{1}{2}$

Solution 3.2 *Click* here *for video.*

Question 3.3 (2023 AQA A-Level Paper 1 Q13) *The function* f *is defined by:*

$$f(x) = \arccos(x),\ 0 \leq x \leq a$$

The curve with equation $y = f(x)$ *is seen below:*



a) *Find the value of* a

*b) On the diagram above, sketch the curve of $y = \cos(x)$ **and** the line $y = x$ both for $0 \leq x \leq \frac{\pi}{2}$*

c) Explain why the solution to the equation $x - \cos(x) = 0$ must also be a solution to the equation $\cos(x) = \arccos(x)$

Solution 3.3 *Click here for video.*

Question 3.4 (2019 Maths Paper 1 Q10) *The volume of a spherical bubble is increasing at a constant rate. Show that the rate of increase of the radius, r, of the bubble is inversely proportional to r^2. The volume of a sphere is $\frac{4}{3}\pi r^3$*

Solution 3.4 *Click here for video*

Question 3.5 (TMUA 2023 Paper 1 Q11) *It is given that $f(x) = x^2 - 6x$. The curves $y = f(kx)$ and $y = f(x - c)$ have the same minimum point, where $k > 0$ and $c > 0$. Find an expression for k in terms of c.*

Solution 3.5 *Click here for video*

Question 3.6 (2019 Further Maths Paper 1 Q2) *The first two nonzero terms of the Maclaurin series expansion of $f(x)$ are x and $-\frac{1}{2}x^3$. Which one of the following could be $f(x)$?*

$$xe^{\frac{1}{2}x^2} \qquad \tfrac{1}{2}\sin(2x) \qquad x\cos(x) \qquad (1+x^3)^{-\frac{1}{2}} \tag{2}$$

Solution 3.6 *Click here for video.*

Question 3.7 *Solve $\int \sin^3(x)\mathrm{d}x$*

Solution 3.7 *This is a tricky one, and may need to try a few approaches before finding one that works. Key to notice is we can split the $\sin^3$ into $\sin$, and $\sin^2$, and then use trigonometric identities and u-substitution with $u = \cos x$:*

$$\sin^3 x = \sin x\,(1 - \cos^2 x).$$

Let $u = \cos x$, so $\mathrm{d}u = -\sin x\,\mathrm{d}x$ and hence $-\mathrm{d}u = \sin x\,\mathrm{d}x$. Then

$$\begin{aligned}
\int \sin^3 x\,\mathrm{d}x &= \int \sin x\,(1 - \cos^2 x)\,\mathrm{d}x \\
&= \int (1 - u^2)\,(-\mathrm{d}u) \\
&= \int (u^2 - 1)\,\mathrm{d}u \\
&= \frac{u^3}{3} - u + C \\
&= \frac{\cos^3 x}{3} - \cos x + C.
\end{aligned}$$

A quick check: differentiating $\frac{\cos^3 x}{3} - \cos x$ gives

$$\big(\cos^2 x(-\sin x)\big) - (-\sin x) = \sin x\,(1 - \cos^2 x) = \sin^3 x.$$

Question 3.8 *June 2024, AQA Further Maths, Paper 1 Q11: Find $\frac{\mathrm{d}}{\mathrm{d}x}\big(x^2\arctan(x)\big)$ and hence find $\int 2x\arctan(x)\mathrm{d}x$.*

Solution 3.8 *Use the product rule and $\arctan$ derivative identity to deduce: $\frac{\mathrm{d}}{\mathrm{d}x}(x^2\arctan(x)) = 2x\arctan(x) + \frac{x^2}{x^2+1}$.*

Therefore,

$$x^2\arctan(x) + C = \int 2x\arctan(x) + \frac{x^2}{x^2+1}\mathrm{d}x$$

So, using the annoying trick of adding $0 = +1 - 1$ and splitting the fraction:

$$\begin{aligned}
\int 2x\arctan(x)\mathrm{d}x &= x^2\arctan(x) - \int \frac{x^2}{x^2+1}\mathrm{d}x + C \\
&= x^2\arctan(x) - \int \frac{x^2+1-1}{x^2+1}\mathrm{d}x + C \\
&= x^2\arctan(x) - \int \frac{x^2+1}{x^2+1}\mathrm{d}x + \int \frac{1}{x^2+1}\mathrm{d}x + C \\
&= x^2\arctan(x) - x + \arctan(x) + C
\end{aligned}$$

Question 3.9 (2023 Further Maths Paper 1 Q15) *Find the general solution of the differential equation,*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = \cos 2x + 5x$$

Solution 3.9
There are different strategies and methods for solving second-order ordinary differential equations, especially in the case of constant coefficients. The most common strategy is to find the general solution for the homogeneous case y_0 and then find a particular solution y_p for the non-homogeneous part. This means,

$$y = y_h + y_p$$

where y_h is the **general** solution of

$$L[y] = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$$

and y_p is **a** solution of the original equation: $L[y_p] = f(x) = \cos 2x + 5x$.

A. **Homogeneous solution** — By *auxiliary (algebraic) equation*
For this method, we write an auxiliary algebraic equation in which we substitute every derivative or order n by r^n. For the present case we have,

$$P(r) = r^2 - 3r^1 - 4r^0 = r^2 - 3r - 4\,.$$

Then, the roots of $P(r) = 0$ will be the exponents for the general solution of the homogeneous equation: $y(x) = \alpha\, e_1^r x + \beta\, e_2^r x$. In the present case, the roots are $\{4, -1\}$, so the general solution of the homogeneous equation is,

$$y_h(x) = C_1 e^{4x} + C_2 e^{-x}\,,$$

where C_1 and C_2 are constants to be determined by the initial conditions.

B. **Particular solution** — By *undetermined coefficients*
In this method, we propose a given form for y_p according to the function we need to match in the non-homogeneous part and find the values of the undetermined coefficients by substituting in the equation. As a rule of thumb:

1. If f is a polynomial of degree m, we set y_p as a polynomial of degree $m-1$.
2. If f is $\sin(bx)$ or $\cos(bx)$, and neither appears in the general solution y_0, then we set $y_p = c_3\cos(bx) + c_4\sin(bx)$.

In this case, using the linearity of L, we can write $y_p = y_{p1} + y_{p2}$, to address each term in $f(x)$. This is,

1) $y_{p1} = c_0 + c_1 x$ to match $5x$, then,

$$\begin{array}{lcl} y'_{p1} = c_1 & \to & y''_{p1} - 3y_{p1} - 4y_{p1} = -3c_1 - 4(c_0 + c_1 x) = 5x \\ & \to & \begin{cases} -4c_1 = 5 \\ 4c_0 - 3c_1 = 0 \end{cases} \quad \to \quad c_0 = 15/16 \end{array}$$

2) $y_{p2} = c_3\cos 2x + c_4\sin 2x$ to match $\cos 2x$. It follows,

$$\begin{array}{lcl} y'_{p2} & = & -2c_3\sin 2x + 2c_4\cos 2x \\ y''_{p2} & = & -4c_3\cos 2x - 4c_4\sin 2x \\ & \to & y''_{p2} - 3y_{p2} - 4y_{p2} = -(6c_4 + 8c_3)\cos 2x + (6c_4 - 8c_3)\sin 2x = \cos 2x \\ & \to & \begin{cases} 6c_4 + 8c_3 = -1 \\ 6c_4 - 8c_3 = 0 \end{cases} \quad \to \quad c_3 = -\frac{2}{25}\,, \quad c_4 = -\frac{3}{50} \end{array}$$

Therefore, a particular solution is,

$$y_p(x) = -\frac{4}{5}x + \frac{15}{16} - \frac{2}{25}\cos 2x - \frac{3}{50}\sin 2x$$

And thus, the general solution is,

$$y(x) = C_1 e^{4x} + C_2 e^{-x} - \frac{4}{5}x + \frac{15}{16} - \frac{2}{25}\cos 2x - \frac{3}{50}\sin 2x\,.$$

Question 3.10 *An octopus is able to catch any fish that swim within a distance of 2m from the octopus's position.*
A fish F swims for point A to point B. The octopus is modelled as a fixed particle at the origin O.
Fish F is modelled as a particle moving in a straight line from A to B.
Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

a) *Use the model to determine whether one or not the octopus is able to catch fish F*

b) *Criticise the model in relation to fish F*

c) *Criticise the model in relation to the octopus*

Solution 3.10 *a) The fish moves along the vector:*

$$\overrightarrow{AB} = \begin{pmatrix} 9-(-3) \\ 4-1 \\ 11-(-7) \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$$

. The fish, F, will then be located at some point along this line:

$$F = \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$$

The vector between the Octopus, O, and Fish is $\overrightarrow{OF}$. As the Octopus is at the origin:

$$\overrightarrow{OF} = \begin{pmatrix} -3+12\lambda-0 \\ 1+3\lambda-0 \\ -7+18\lambda-0 \end{pmatrix} = \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix}$$

The fastest way for the Octopus to get the Fish is to go to the point where $\overrightarrow{OF}$ and $\overrightarrow{AB}$ are perpendicular. Two vectors are perpendicular when their dot product is 0:

$$\begin{aligned} \overrightarrow{OF} \cdot \overrightarrow{AB} &= \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} \\ &= -36+144\lambda+3+9\lambda-126+324\lambda \\ &= 477\lambda - 159 = 0 \\ \Rightarrow \lambda &= \frac{1}{3} \end{aligned}$$

So the Octopus catches the Fish when $\lambda = \frac{1}{3}$. We then find the vector $\overrightarrow{OF}$ by subbing in:

$$\overrightarrow{OF} = \begin{pmatrix} -3+12(\frac{1}{3}) \\ 1+3(\frac{1}{3}) \\ -7+18(\frac{1}{3}) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

This vector has length:

$$|\overrightarrow{OF}| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6} = 2.449 > 2$$

Hence, the shortest distance between the Octopus and the Fish is greater than 2 metres and the Octopus cannot catch it.

---

b) The model assumes the Fish swims in a straight line and does nothing to avoid the Octopus.

---

c) The Octopus is assumed to be a particle and so there is no accounting for the mass. It also is assumed to always start at the origin.

Question 3.11 (2023 TMUA Paper 1 Q4) *Evaluate*

$$\sum_{n=0}^{\infty} \frac{\sin(n\pi+\frac{\pi}{3})}{2^n}$$

a) 0

b) $\frac{1}{3}$

c) $\frac{\sqrt{3}}{3}$

d) $\sqrt{3}$

e) 3

Solution 3.11 *Click here for video.*

Question 3.12 (2023 TMUA Paper 2 Q16) *A sequence is defined by*

$$u_1 = a$$
$$u_2 = b$$
$$u_{n+2} = u_n + u_{n+1}$$

for $n \geq 1$ *and where* a *and* b *are positive integers. The highest common factor of* a *and* b *is 7. Which of the following* ***must*** *be true.*

I) u_{2023} *is a multiple of 7*

II) If u_1 *is not a factor of* u_2*, then* u_1 *is not a factor of* u_n *for any n ¿ 1*

III) The highest common factor of u_1 *and* u_5 *is 7*

a) none of them

b) I only

c) II only

d) III only

e) I and II only

f) I and III only

g) II and III only

h) I, II and III

Solution 3.12 *Click here for video.*

Question 3.13 (2022 TMUA Paper 1 Q5) *The terms* x_n *of a sequence follow the rule:*

$$x_{n+1} = \frac{x_n + p}{x_n + q}$$

where $p, q \in \mathbb{R}$*. Given that* $x_1 = 3, x_2 = 5$ *and* $x_3 = 7$*, find the value of* x_4*.*

A) -5

B) 5

C) $\frac{51}{7}$

D) $\frac{15}{2}$

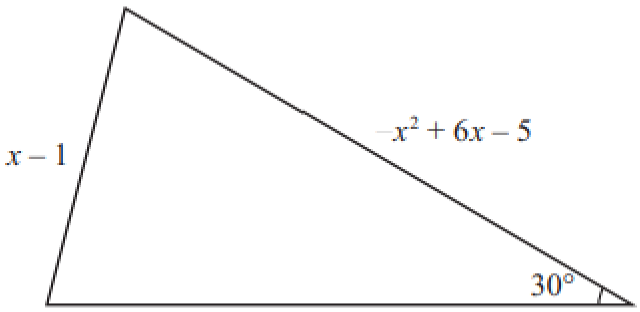
E) $\frac{23}{3}$

F) 9

G) 11

H) 13

Solution 3.13 *Click here for video.*



Question 3.14
Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A) $1 < x < 3$

B) $1 < x < 4$

C) $1 < x < 5$

D) $3 < x < 4$

E) $3 < x < 5$

F) $4 < x < 5$

Note: the sides are $x - 1$ and $-x^2 + 6x - 5$: the minus signs can be difficult to see.

Solution 3.14 *Click here for video*