Interacting Particle Approximations of Feynman-Kac Formulae for Pure Jump Processes

Letizia Angeli

Joint work with: *Adam Johansen*, *Stefan Grosskinsky*

13th November 2020

Motivations

- Interest in studying the dynamical large deviations of stochastic models, e.g.
  - chemical reactions
  - glassy dynamics
  - particle current in lattice gases
  - heat waves in climate models
- Interacting Particle Approximations (IPAs):
  - CLONING ALGORITHM [C. Giardinà et al. (2006), V. Lecomte and J. Tailleur (2007)]
  - MEAN FIELD PARTICLE APPROXIMATIONS [P. Del Moral and L. Miclo (2000), M. Rousset (2006)]

The Setting: Pure Jump Markov Processes

- E, locally compact (Polish) state space
  - e.g. spin systems with $E = \{-1, 1\}^\Lambda$, Λ lattice
  - or exclusion processes with $E = \{0, 1\}^\Lambda$
- $\mathcal{C}_b(E)$, set of bounded continuous functions on E with norm $||f|| := \sup_{x \in E} |f(x)|$
- X_t, continuous-time **pure jump** Markov process on E, with initial distribution μ_0. Denote:
  - $\lambda(x) \in \mathcal{C}_b(E)$, $\lambda(x) > 0$, escape rate at state $x \in E$.
  - $p(x, dy)$, the probability of jumping from x to y, when a jump occurs
  - $W(x, dy)$, the overall transition rate, i.e.

$$W(x, dy) := \lambda(x) \cdot p(x, dy)$$

Infinitesimal Description of Pure Jump Markov Processes

$\mathcal{L}$ - infinitesimal generator of X_t

$$\frac{d}{dt}\mathbb{E}_x[f(X_t)] = \mathbb{E}_x[\mathcal{L}f(X_t)]$$

where $f \in \mathcal{C}_b(E)$ observable.

For pure jump Markov processes, we can write

$$\mathcal{L}f(x) = \int_E W(x, dy)\left(f(y) - f(x)\right), \quad f \in \mathcal{C}_b(E)$$

where $W(x, dy)$ is the overall transition rate.

Feynman-Kac Measures

Let X_t be a jump process on E with generator $\mathcal{L}$ and initial distribution μ_0, and let $\mathcal{V} \in \mathcal{C}_b(E)$ be a potential function.

Feynman-Kac measures associated to $(X_t, \mathcal{V})$

$$\nu_t(f) := \mathbb{E}_{\mu_0}\Big[f(X_t)\exp\Big(\int_0^t \mathcal{V}(X_s)ds\Big)\Big], \quad \mu_t(f) := \frac{\nu_t(f)}{\nu_t(1)} .$$

Feynman-Kac Measures

Let X_t be a jump process on E with generator $\mathcal{L}$ and initial distribution μ_0, and let $\mathcal{V} \in \mathcal{C}_b(E)$ be a potential function.

Feynman-Kac measures associated to $(X_t, \mathcal{V})$

$$\nu_t(f) := \mathbb{E}_{\mu_0}\Big[f(X_t) \exp\Big(\int_0^t \mathcal{V}(X_s) ds\Big)\Big], \quad \mu_t(f) := \frac{\nu_t(f)}{\nu_t(1)} .$$

Remark. We can write ν_t in terms of μ_t,

$$\nu_t(f) = \mu_t(f) \cdot \exp\Big(\int_0^t \mu_s(\mathcal{V}) ds\Big)$$

Feynman-Kac Measures

Let X_t be a jump process on E with generator $\mathcal{L}$ and initial distribution μ_0, and let $\mathcal{V} \in \mathcal{C}_b(E)$ be a potential function.

Feynman-Kac measures associated to $(X_t, \mathcal{V})$

$$\nu_t(f) := \mathbb{E}_{\mu_0}\Big[f(X_t)\exp\Big(\int_0^t \mathcal{V}(X_s)ds\Big)\Big], \quad \mu_t(f) := \frac{\nu_t(f)}{\nu_t(1)} .$$

Remark. We can write ν_t in terms of μ_t,

$$\nu_t(f) = \mu_t(f) \cdot \exp\Big(\int_0^t \mu_s(\mathcal{V})ds\Big)$$

Evolution Equation

$$\frac{d}{dt}\mu_t(f) = \mu_t\Big(\mathcal{L}f + \mathcal{V}f - \mu_t(\mathcal{V})\, f\Big)$$

McKean Interpretation

There exists a (non-unique) family of non-linear probability generators $(\widetilde{\mathcal{L}}_\mu)$, $\mu \in \mathcal{P}(E)$, s.t.

$$\mu(\widetilde{\mathcal{L}}_\mu(f)) = \mu(\mathcal{V}f) - \mu(\mathcal{V}) \cdot \mu(f) .$$

Thus $\mu_t = Law(\overline{X}_t)$, where $\overline{X}_t$ stochastic process associated to the generator $\overline{\mathcal{L}}_{\mu_t} := \mathcal{L} + \widetilde{\mathcal{L}}_{\mu_t}$.

McKean Interpretation

There exists a (non-unique) family of non-linear probability generators $\left(\widetilde{\mathcal{L}}_\mu\right)$, $\mu \in \mathcal{P}(E)$, s.t.

$$\mu(\widetilde{\mathcal{L}}_\mu(f)) = \mu(\mathcal{V}f) - \mu(\mathcal{V}) \cdot \mu(f) .$$

Thus $\mu_t = Law(\overline{X}_t)$, where $\overline{X}_t$ stochastic process associated to the generator $\overline{\mathcal{L}}_{\mu_t} := \mathcal{L} + \widetilde{\mathcal{L}}_{\mu_t}$.

Possible constructions $\widetilde{\mathcal{L}}_\mu$ are in the form

$$\widetilde{\mathcal{L}}_\mu f(x) = \int_E \widetilde{W}(x,y)\big(f(y) - f(x)\big)\mu(dy) ,$$

with

- $\widetilde{W}(x,y) = \big(\mathcal{V}(x) - c\big)^- + \big(\mathcal{V}(y) - c\big)^+ , \; c \in \mathbb{R}$
- $\widetilde{W}(x,\, y) = \big(\mathcal{V}(y) - \mathcal{V}(x)\big)^+$

Interacting Particle Approximations

An interacting particle approximation is a family of N-particle systems $\underline{X}_t^N := (X_t^1, \ldots, X_t^N) \in E^N$, $N \in \mathbb{N}$, with empirical distribution

$$\mu_t^N := m(\underline{X}_t^N) = \frac{1}{N} \sum_i \delta_{X_t^i}.$$

Interacting Particle Approximations

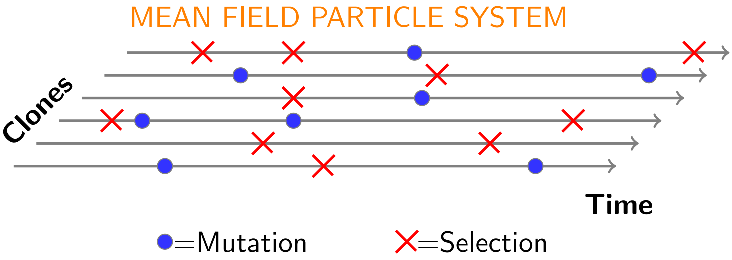
An interacting particle approximation is a family of N-particle systems $\underline{X}_t^N := (X_t^1, \ldots, X_t^N) \in E^N$, $N \in \mathbb{N}$, with empirical distribution

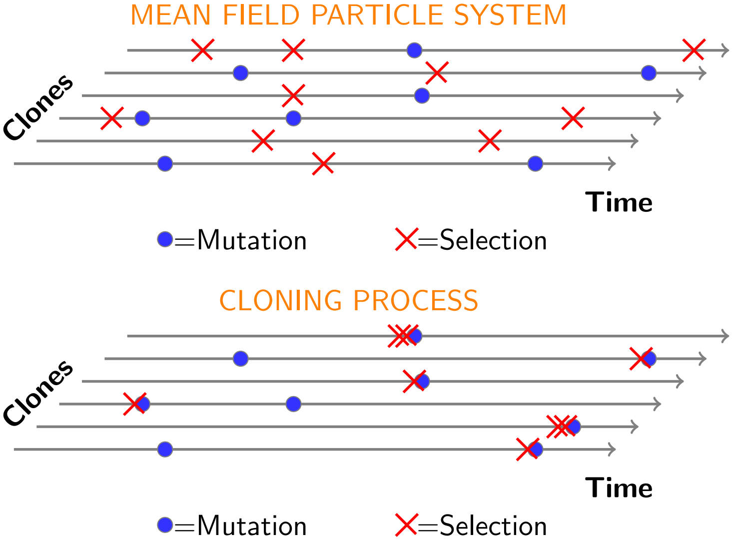
$$\mu_t^N := m(\underline{X}_t^N) = \frac{1}{N} \sum_i \delta_{X_t^i}.$$

Find sufficient conditions s.t. $\mu_T^N(f) \sim \mu_T(f)$ and, in particular, s.t. for any $p \geq 2$, there exists $c_p > 0$ (indep. of N or T)

$$\sup_{T \geq 0} \mathbb{E}\left[\left|\mu_T^N(f) - \mu_T(f)\right|^p\right]^{1/p} \leq \frac{c_p \|f\|}{N^{1/2}},$$

for any $f \in \mathcal{C}_b(E)$ and $N \in \mathbb{N}$ large enough.


MEAN FIELD PARTICLE SYSTEM
Clones
Time
=Mutation
=Selection



MEAN FIELD PARTICLE SYSTEM
Clones
Time
=Mutation
=Selection
CLONING PROCESS
Clones
Time
=Mutation
=Selection

Assumptions

Denoting by L^N the generator, the carré du champ is

$$\Gamma_{L^N}(F,F)(\underline{x}) := L^N(F^2)(\underline{x}) - 2F(\underline{x})\, L^N(F)(\underline{x}) \,.$$

Conditions

(0) ASYMPTOTIC STABILITY of μ_t:

$$\exists\, \rho \in (0,1) \quad \text{s.t.} \quad \big|\mu_t(f) - \mu_\infty(f)\big| \le c\|f\|\,\rho^t$$

independently of the initial distribution μ_0 ;

(1) $X_0^1, \ldots, X_0^N \sim \mu_0$ i.i.d. random variables;

Assumptions

Denoting by L^N the generator, the carré du champ is

$$\Gamma_{L^N}(F,F)(\underline{x}) := L^N(F^2)(\underline{x}) - 2F(\underline{x})\, L^N(F)(\underline{x}) \;.$$

Conditions

(0) ASYMPTOTIC STABILITY of μ_t:

$$\exists\, \rho \in (0,1) \quad \text{s.t.} \quad \left|\mu_t(f) - \mu_\infty(f)\right| \leq c\|f\|\, \rho^t$$

independently of the initial distribution μ_0 ;

(1) $X_0^1, \ldots, X_0^N \sim \mu_0$ i.i.d. random variables;

Moreover, for $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$,

(2) $L^N(F)(\underline{x}) = m(\underline{x})\big(\overline{\mathcal{L}}_{m(\cdot)}(f)\big)$;

Assumptions

Denoting by L^N the generator, the carré du champ is

$$\Gamma_{L^N}(F,F)(\underline{x}) := L^N(F^2)(\underline{x}) - 2F(\underline{x})\, L^N(F)(\underline{x}) \,.$$

Conditions

(0) ASYMPTOTIC STABILITY of μ_t:

$$\exists\, \rho \in (0,1) \quad \text{s.t.} \quad \left|\mu_t(f) - \mu_\infty(f)\right| \leq c\|f\|\, \rho^t$$

independently of the initial distribution μ_0 ;

(1) $X_0^1, \ldots, X_0^N \sim \mu_0$ i.i.d. random variables;

Moreover, for $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$,

(2) $L^N(F)(\underline{x}) = m(\underline{x})\big(\overline{\mathcal{L}}_{m(\cdot)}(f)\big)$;

(3) $\Gamma_{L^N}(F,F)(\underline{x}) \leq \frac{c\|f\|^2}{N}$;

Assumptions

Denoting by L^N the generator, the carré du champ is

$$\Gamma_{L^N}(F,F)(\underline{x}) := L^N(F^2)(\underline{x}) - 2F(\underline{x})\, L^N(F)(\underline{x}) \ .$$

Conditions

(0) ASYMPTOTIC STABILITY of μ_t:

$$\exists\, \rho \in (0,1) \quad \text{s.t.} \quad \big|\mu_t(f) - \mu_\infty(f)\big| \leq c\|f\|\, \rho^t$$

independently of the initial distribution μ_0 ;

(1) $X_0^1, \ldots, X_0^N \sim \mu_0$ i.i.d. random variables;

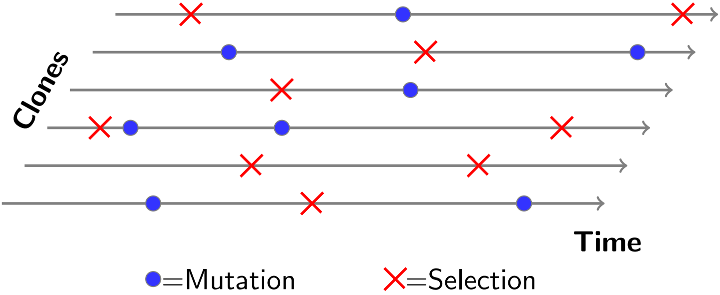
Moreover, for $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$,

(2) $L^N(F)(\underline{x}) = m(\underline{x})\big(\overline{\mathcal{L}}_{m(\cdot)}(f)\big)$;

(3) $\Gamma_{L^N}(F,F)(\underline{x}) \leq \frac{c\|f\|^2}{N}$;

(4) $\sup_{t\geq 0} |\{i \in 1,\ldots,N \,|\, X_t^i \neq X_{t-}^i\}| \leq K$ a.s., K indep. of N.

Mean Field Particle Approximations



For $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$,

$$L^N(F)(\underline{x}) = \sum_{i=1}^{N} \left(\mathcal{L} + \widetilde{\mathcal{L}}_{m(\underline{x})}\right)(f)(x_i) = m(\underline{x})\left(\overline{\mathcal{L}}_{m(\cdot)}(f)\right)$$

$$\Gamma_{L^N}(F,F)(\underline{x}) = \frac{1}{N}\, m(\underline{x})\left(\Gamma_{\mathcal{L}+\widetilde{\mathcal{L}}_{m(\underline{x})}}(f,f)\right) \leq \frac{c\|f\|^2}{N}$$

Cloning Algorithm (1/2)

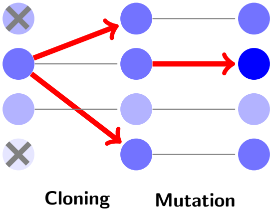


Figure: Illustration of a single cloning-mutation event.

$$L^N(F)(\underline{x}) = \sum_{i=1}^{N} \lambda(x_i) \int_E p(x_i,\, dy) \sum_{A \in \mathcal{N}} \pi_{\underline{x}}(x_i, A) \cdot \left(F(\underline{x}^{A, x_i;\, i, y}) - F(\underline{x}) \right)$$

Cloning Algorithm (2/2)

Choosing $\pi_{\underline{x}}(x_i, A)$ s.t.

$$\sum_{A|j\in A} \pi_{\underline{x}}(x_i, A) = \frac{\widetilde{W}(x_j, x_i)}{N \cdot \lambda(x_i)} ,$$

for N large enough, then

$$L^N(F)(\underline{x}) = m(\underline{x})\big(\overline{\mathcal{L}}_{m(\cdot)}(f)\big) ,$$

for $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$.

Cloning Algorithm (2/2)

Choosing $\pi_{\underline{x}}(x_i, A)$ s.t.

$$\sum_{A|j\in A} \pi_{\underline{x}}(x_i, A) = \frac{\widetilde{W}(x_j, x_i)}{N \cdot \lambda(x_i)} ,$$

for N large enough, then

$$L^N(F)(\underline{x}) = m(\underline{x})\big(\overline{\mathcal{L}}_{m(\cdot)}(f)\big) ,$$

for $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$.

Moreover, if

$$\sum_{A|j,k\in A} \pi_{\underline{x}}(x_i, A) \leq \frac{C}{N^2} ,$$

then

$$\Gamma_{L^N}(F, F)(\underline{x}) \leq \frac{c\|f\|^2}{N} .$$

Assumptions

Denoting by L^N the generator, the carré du champ is

$$\Gamma_{L^N}(F,F)(\underline{x}) := L^N(F^2)(\underline{x}) - 2F(\underline{x})\,L^N(F)(\underline{x})\ .$$

Conditions

(0) ASYMPTOTIC STABILITY of μ_t:

$$\exists\,\rho \in (0,1) \quad \text{s.t.} \quad \left|\mu_t(f) - \mu_\infty(f)\right| \le c\|f\|\,\rho^t$$

independently of the initial distribution μ_0 ;

(1) $X_0^1, \ldots, X_0^N \sim \mu_0$ i.i.d. random variables;

Moreover, for $F(\underline{x}) = m(\underline{x})(f)$ with $f \in \mathcal{C}_b(E)$,

(2) $L^N(F)(\underline{x}) = m(\underline{x})\big(\overline{\mathcal{L}}_{m(\cdot)}(f)\big)$;

(3) $\Gamma_{L^N}(F,F)(\underline{x}) \le \frac{c\|f\|^2}{N}$;

(4) $\sup_{t\ge 0} |\{i \in 1,\ldots,N \,|\, X_t^i \ne X_{t-}^i\}| \le K$ a.s., K indep. of N.

Martingale interpretation of conditions (2)-(4)

$$\mathcal{M}_t^N(\varphi_\cdot) := \mu_t^N(\varphi_t) - \mu_0^N(\varphi_0) - \int_0^t \mu_s^N\big(\partial_s\varphi_s + \overline{\mathcal{L}}_{\mu_s^N}(\varphi_s)\big)\ ds$$

is a martingale with

$$\langle \mathcal{M}_\cdot^N(\varphi_\cdot)\rangle_0^t \ \leq\ \int_0^t \frac{c\,\|\varphi_s\|^2}{N}\, ds\ ; \qquad \big|\Delta\mathcal{M}_t^N(\varphi_\cdot)\big| \leq \frac{2K\|\varphi_t\|}{N}\ .$$

Technical Lemma

Let $\mathcal{N}$ be a square-integrable martingale with $\sup_t |\Delta\mathcal{N}_t| \leq a$. Then,

$$\sup_{t\leq T} \mathbb{E}\big[\big|\mathcal{N}_t\big|^{2^q}\big] \ \leq\ c_q \sum_{k=1}^{q} a^{2^q-2^k}\mathbb{E}\big[\big(\langle\mathcal{N}\rangle_T\big)^{2^{k-1}}\big]\ , \quad \forall q\in\mathbb{N}\ .$$

Time-uniform bias and L^p Error Estimates

THEOREM

Under Assumptions (0)-(4), there exists $c' > 0$ s.t.

$$\sup_{T\geq 0}\left|\mathbb{E}\left[\mu_T^N(f)-\mu_T(f)\right]\right| \leq \frac{c'\|f\|}{N},$$

and, for any $p \geq 2$, there exists $c_p > 0$ s.t.

$$\sup_{T\geq 0}\mathbb{E}\left[\left(\mu_T^N(f)-\mu_T(f)\right)^p\right]^{1/p} \leq \frac{c_p\|f\|}{N^{1/2}},$$

for any $f \in \mathcal{C}_b(E)$ and $N \in \mathbb{N}$ large enough.

References

L. Angeli, S. Grosskinsky, and A. M. Johansen.
Limit theorems for cloning algorithms.
arXiv preprint arXiv:1902.00509, 2019.

L. Angeli, S. Grosskinsky, A. M. Johansen, and A. Pizzoferrato.
Rare event simulation for stochastic dynamics in continuous time.
Journal of Statistical Physics, 176(5):1185–1210, 2019.

P. Del Moral and L. Miclo.
Branching and interacting particle systems approximations of Feynman-Kac formulae with applications to non-linear filtering.
In *Seminaire de probabilites XXXIV*, pages 1–145. Springer, 2000.

C. Giardina, J. Kurchan, and L. Peliti.
Direct evaluation of large-deviation functions.
Physical review letters, 96(12):120603, 2006.

M. Rousset.
On the control of an interacting particle estimation of Schrödinger ground states.
SIAM journal on mathematical analysis, 38(3):824–844, 2006.