Interacting Particle Approximations of Feynman-Kac Formulae for Pure Jump Processes

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13th November 2020
Motivations

- Interest in studying the dynamical large deviations of stochastic models, e.g.
  - chemical reactions
  - glassy dynamics
  - particle current in lattice gases
  - heat waves in climate models

- Interacting Particle Approximations (IPAs):
  - CLONING ALGORITHM [C. Giardinà et al. (2006), V. Lecomte and J. Tailleur (2007)]
The Setting: Pure Jump Markov Processes

- $E$, locally compact (Polish) state space
  - e.g. spin systems with $E = \{-1, 1\}^\Lambda$, $\Lambda$ lattice
  - or exclusion processes with $E = \{0, 1\}^\Lambda$

- $C_b(E)$, set of bounded continuous functions on $E$ with norm
  $\|f\| := \sup_{x \in E} |f(x)|$

- $X_t$, continuous-time pure jump Markov process on $E$, with initial distribution $\mu_0$. Denote:
  - $\lambda(x) \in C_b(E)$, $\lambda(x) > 0$, escape rate at state $x \in E$.
  - $p(x, dy)$, the probability of jumping from $x$ to $y$, when a jump occurs
  - $W(x, dy)$, the overall transition rate, i.e.
    \[ W(x, dy) := \lambda(x) \cdot p(x, dy) \]
Infinitesimal Description of Pure Jump Markov Processes

\( \mathcal{L} \) - infinitesimal generator of \( X_t \)

\[
\frac{d}{dt} \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[\mathcal{L}f(X_t)]
\]

where \( f \in C_b(E) \) observable.

For pure jump Markov processes, we can write

\[
\mathcal{L}f(x) = \int_E W(x, dy) (f(y) - f(x)), \quad f \in C_b(E)
\]

where \( W(x, dy) \) is the overall transition rate.
Feynman-Kac Measures

Let $X_t$ be a jump process on $E$ with generator $\mathcal{L}$ and initial distribution $\mu_0$, and let $\mathcal{V} \in \mathcal{C}_b(E)$ be a potential function.

**Feynman-Kac measures associated to $(X_t, \mathcal{V})$**

$$\nu_t(f) := \mathbb{E}_{\mu_0} \left[ f(X_t) \exp \left( \int_0^t \mathcal{V}(X_s) ds \right) \right], \quad \mu_t(f) := \frac{\nu_t(f)}{\nu_t(1)}.$$
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Remark. We can write $\nu_t$ in terms of $\mu_t$,

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Evolution Equation

$$\frac{d}{dt} \mu_t(f) = \mu_t \left( \mathcal{L}f + \mathcal{V}f - \mu_t(\mathcal{V}) f \right)$$
McKean Interpretation

There exists a (non-unique) family of non-linear probability generators $(\tilde{\mathcal{L}}_\mu)$, $\mu \in \mathcal{P}(E)$, s.t.

$$\mu(\tilde{\mathcal{L}}_\mu(f)) = \mu(\mathcal{V}f) - \mu(\mathcal{V}) \cdot \mu(f).$$

Thus $\mu_t = \text{Law}(\overline{X}_t)$, where $\overline{X}_t$ stochastic process associated to the generator $\overline{\mathcal{L}}_{\mu_t} := \mathcal{L} + \tilde{\mathcal{L}}_{\mu_t}$. 
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Possible constructions \(\tilde{\mathcal{L}}_\mu\) are in the form

\[
\tilde{\mathcal{L}}_\mu f(x) = \int_E \tilde{\mathcal{W}}(x, y)(f(y) - f(x)) \mu(dy) ,
\]

with

\[
\tilde{\mathcal{W}}(x, y) = (\mathcal{V}(x) - c)^- + (\mathcal{V}(y) - c)^+ , \ c \in \mathbb{R}
\]

\[
\tilde{\mathcal{W}}(x, y) = (\mathcal{V}(y) - \mathcal{V}(x))^+ .
\]
An interacting particle approximation is a family of $N$-particle systems $X^N_t := (X^1_t, \ldots, X^N_t) \in E^N$, $N \in \mathbb{N}$, with empirical distribution

$$\mu^N_t := m(X^N_t) = \frac{1}{N} \sum_i \delta_{X^i_t}.$$
Interacting Particle Approximations

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Find sufficient conditions s.t. $\mu^N_T(f) \sim \mu_T(f)$ and, in particular, s.t. for any $p \geq 2$, there exists $c_p > 0$ (indep. of $N$ or $T$)

$$\sup_{T \geq 0} \mathbb{E} \left[ |\mu^N_T(f) - \mu_T(f)|^p \right]^{1/p} \leq \frac{c_p \|f\|}{N^{1/2}},$$

for any $f \in C_b(E)$ and $N \in \mathbb{N}$ large enough.
MEAN FIELD PARTICLE SYSTEM

Clones

Time

● = Mutation  ❌ = Selection
MEAN FIELD PARTICLE SYSTEM

CLONING PROCESS
Assumptions

Denoting by $L^N$ the generator, the carré du champ is

$$\Gamma_{L^N}(F, F)(\bar{x}) := L^N(F^2)(\bar{x}) - 2F(\bar{x})L^N(F)(\bar{x}).$$

Conditions

(0) **ASYMPTOTIC STABILITY** of $\mu_t$:

$$\exists \rho \in (0, 1) \text{ s.t. } |\mu_t(f) - \mu_\infty(f)| \leq c \|f\| \rho^t$$

independently of the initial distribution $\mu_0$ ;

(1) $X_0^1, \ldots, X_0^N \sim \mu_0$ i.i.d. random variables;
Assumptions

Denoting by $L^N$ the generator, the carré du champ is

$$\Gamma_{L^N}(F, F)(x) := L^N(F^2)(x) - 2F(x) L^N(F)(x).$$

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Moreover, for $F(x) = m(x)(f)$ with $f \in C_b(E)$,

(2) $L^N(F)(x) = m(x)(\mathcal{L}m(\cdot)(f))$;
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(2) $L^N(F)(\bar{x}) = m(\bar{x})(\overline{L} m(\cdot)(f))$;

(3) $\Gamma_{L^N}(F, F)(\bar{x}) \leq \frac{c\|f\|^2}{N}$;
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(4) $\sup_{t \geq 0} |\{i \in 1, \ldots, N \mid X_t^i \neq X_{t-}^i\}| \leq K$ a.s., $K$ indep. of $N$.‌
Mean Field Particle Approximations

For $F(x) = m(x)(f)$ with $f \in \mathcal{C}_b(E)$,

$$L^N(F)(x) = \sum_{i=1}^{N} (\mathcal{L} + \tilde{\mathcal{L}}_{m(x)})(f)(x_i) = m(x)(\overline{\mathcal{L}}_{m(\cdot)}(f))$$

$$\Gamma_{LN}(F, F)(x) = \frac{1}{N} m(x)(\Gamma_{\mathcal{L} + \tilde{\mathcal{L}}_{m(x)}}(f, f)) \leq \frac{c\|f\|^2}{N}$$
Cloning Algorithm (1/2)

**Figure:** Illustration of a single cloning-mutation event.

\[ L^N(F)(\mathbf{x}) = \sum_{i=1}^{N} \lambda(x_i) \int_E p(x_i, dy) \sum_{A \in \mathcal{N}} \pi_x(x_i, A) \cdot (F(x^A_{x_i}; i, y) - F(\mathbf{x})) \]
Cloning Algorithm (2/2)

Choosing $\pi_{\mathbf{x}}(x_i, A)$ s.t.

$$\sum_{A \mid j \in A} \pi_{\mathbf{x}}(x_i, A) = \frac{\tilde{W}(x_j, x_i)}{N \cdot \lambda(x_i)},$$

for $N$ large enough, then

$$L^N(F)(\mathbf{x}) = m(\mathbf{x})(\overline{\mathcal{L}}_{m(\cdot)}(f)),$$

for $F(\mathbf{x}) = m(\mathbf{x})(f)$ with $f \in \mathcal{C}_b(E)$. 
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$$

for $N$ large enough, then

$$
L_N^N(F)(x) = m(x)(\mathcal{L}_{m(\cdot)}(f)),
$$

for $F(x) = m(x)(f)$ with $f \in C_b(E)$.

Moreover, if

$$
\sum_{A|j,k \in A} \pi_{x_i}(x_i, A) \leq \frac{C}{N^2},
$$

then

$$
\Gamma_{L_N}(F, F)(x) \leq \frac{c\|f\|^2}{N}.
$$
Assumptions

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Martingale interpretation of conditions (2)-(4)

\[ M^N_t(\phi.) := \mu^N_t(\phi_t) - \mu^N_0(\phi_0) - \int_0^t \mu^N_s(\partial_s \phi_s + \overline{L}_{\mu^N_s}(\phi_s)) \, ds \]

is a martingale with

\[ \langle M^N(\phi.) \rangle_0^t \leq \int_0^t c \frac{\|\phi_s\|^2}{\mathcal{N}} \, ds; \quad |\Delta M^N_t(\phi.)| \leq \frac{2K \|\phi_t\|}{\mathcal{N}}. \]

Technical Lemma

Let \( \mathcal{N} \) be a square-integrable martingale with \( \sup_t |\Delta \mathcal{N}_t| \leq a \). Then,

\[ \sup_{t \leq T} \mathbb{E}[|\mathcal{N}_t|^{2q}] \leq c_q \sum_{k=1}^q a^{2^q - 2^k} \mathbb{E}[\left(\langle \mathcal{N} \rangle_T\right)^{2^{k-1}}], \quad \forall q \in \mathbb{N}. \]
THEOREM
Under Assumptions (0)-(4), there exists $c'>0$ s.t.

$$\sup_{T \geq 0} \left| \mathbb{E} \left[ \mu_N^T(f) - \mu_T(f) \right] \right| \leq \frac{c' \|f\|}{N},$$

and, for any $p \geq 2$, there exists $c_p > 0$ s.t.

$$\sup_{T \geq 0} \mathbb{E} \left[ \left( \mu_N^T(f) - \mu_T(f) \right)^p \right]^{1/p} \leq \frac{c_p \|f\|}{N^{1/2}},$$

for any $f \in C_b(E)$ and $N \in \mathbb{N}$ large enough.
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