Network inference in a stochastic multi-population neural mass model via ABC

Algorithm Seminar 2023

WARWICK

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Can we infer the connectivity structures of brain regions before and during epileptic seizure? Do they differ?

Algorithm Semina

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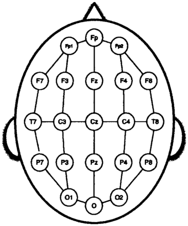
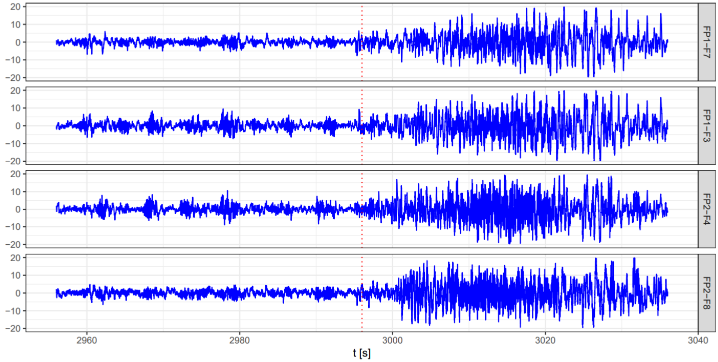
EEG recordings





EEG recordings of a 11 year old female patient.

A few questions to answer

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  How to simulate EEG recordings from the chosen model?

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- Modelling: How to model this?
- Simulation/Numerics:
  How to simulate EEG recordings from the chosen model?
- Statistics:
  How to infer such network structure?

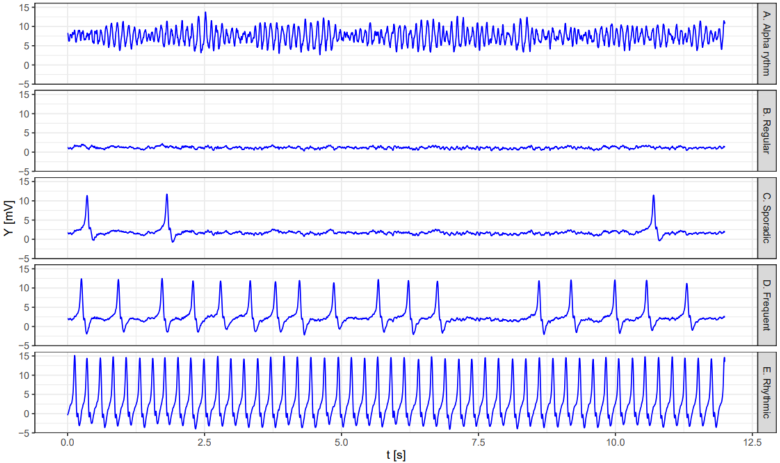
Let's look at one neural population

Model: 6-dim Jansen-and-Rit NMM (Hamiltonian-type SDE)

$$
\begin{aligned}
dX_1(t) &= X_4(t)dt \\
dX_2(t) &= X_5(t)dt \\
dX_3(t) &= X_6(t)dt \\
dX_4(t) &= \left[Aa(\mathrm{sig}(X_2(t) - X_3(t))) - 2aX_4(t) - a^2X_1(t)\right] dt + \bar{\varepsilon} dW_4(t) \\
dX_5(t) &= \left[Aa(\mu + C_2\mathrm{sig}(C_1X_1(t))) - 2aX_5(t) - a^2X_2(t)\right] dt + \sigma dW_5(t) \\
dX_6(t) &= \left[BbC_4\mathrm{sig}(C_3X_1(t)) - 2bX_6(t) - b^2X_3(t)\right] dt + \tilde{\varepsilon} dW_6(t),
\end{aligned}
\tag{1}
$$

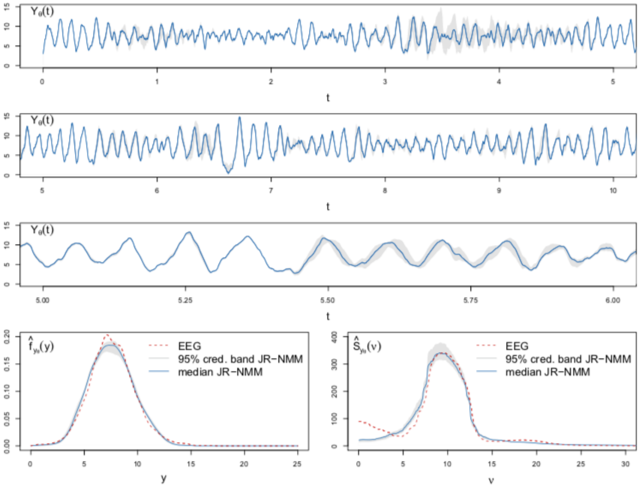
with fixed $\bar{\varepsilon}, \tilde{\varepsilon} \ll \sigma$ and unknown parameters μ, C, σ.

What can a single sJR-NMM do?



Different activities obtained by modifying the excitation-inhibition-ratio A/B

It succesfully fits single EEG recording[1]



[1] Buckwar, Tamborrino, Tubikanec. Spectral density-based and measure-preserving ABC for partially observed diffusion processes. An illustration on Hamiltonian SDEs. Stat. Comput. 30 (3), 627-648, 2020.

Modelling of N copuled neural populations

- N populations of neural mass models $\Rightarrow$ 6N-SDE
- Each population k follows a sJR-NMM with

$$dX_5(t) = \left[Aa(\mu + C_2\mathrm{sig}(C_1X_1(t))) - 2aX_5(t) - a^2X_2(t)\right]dt + \sigma dW_5(t)$$

$$\Rightarrow dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2,k}\mathrm{sig}\left(C_{1,k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk}K_{jk}X_1^j(t)\right) - 2a_kX_5^k(t) - a_k^2X_2^k(t)\right]dt + \sigma_k dW_5^k(t$$

with

* $\rho_{jk} \in \{0,1\}$ modelling the directed coupling from the jth to kth pop

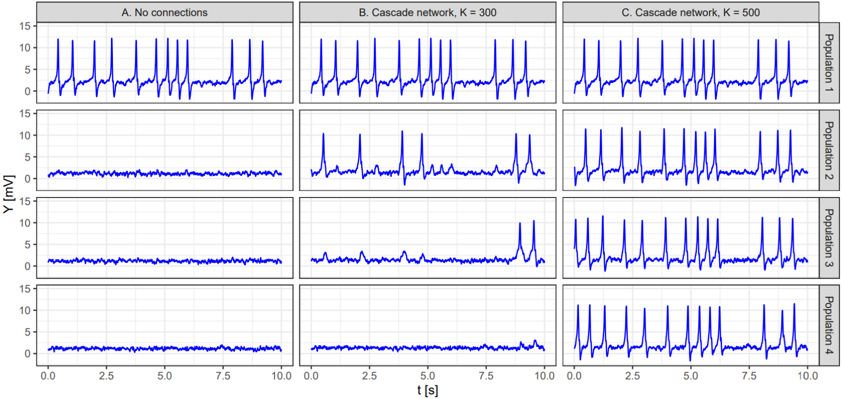
* $K_{jk} > 0$ modelling the coupling strenght.

- N-dimensional observed *output*

$$Y(t) := (Y^1(t), \ldots, Y^N(t))^\top = (X_2^1(t) - X_3^1(t), \ldots, X_2^N(t) - X_3^N(t))^\top, \quad t \in [0, T].$$

- Now the excitation-inhibitiona ratio A/B, ρ_{ij} and K_{ij} play a crucial role

Example: Cascade network - 4 populations, 1 active



Population 1: Setting D: Frequent spiking.
Left columns: $\rho_{kj} = 0$.
Center and Right columns: $\rho_{12} = \rho_{23} = \rho_{34} = 1$, $K_{ii+1} = 300$ (C) vs 500 (R).

Formulation as stochastic Hamiltonian-type system

Each kth population can be written as

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} \nabla_P H_k\left(Q^k(t), P^k(t)\right) \\ -\nabla_Q H_k\left(Q^k(t), P^k(t)\right) - 2\Gamma_k P^k(t) + G_k(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_3 \\ \Sigma_k \end{pmatrix} dW^k(t),$$

with $H_k : \mathbb{R}^6 \to \mathbb{R}_0^+$ given by

$$H_k(Q^k, P^k) := \frac{1}{2}\left(\left\|P^k\right\|^2 + \left\|\Gamma_k Q^k\right\|^2\right),$$

with:
- gradients $\nabla_P H_k\left(Q^k(t), P^k(t)\right) = P^k(t)$ and $\nabla_Q H_k\left(Q^k(t), P^k(t)\right) = \Gamma_k^2 Q^k(t)$
- 3×3-dimensional diagonal matrix $\Gamma_k = \mathrm{diag}[a_k, a_k, b_k]$. and $G_k : \mathbb{R}^{3N} \to \mathbb{R}^3$ given by

$$G_k(Q(t)) = \begin{pmatrix} A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) \\ A_k a_k\left(\mu_k + C_{2,k}\mathrm{sig}\left(C_{1,k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) \\ B_k b_k C_{4,k}\mathrm{sig}(C_{3,k}X_1^k(t)) \end{pmatrix},$$

Formulation as stochastic Hamiltonian-type system

Putting everything together

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t),$$

with

$$\begin{aligned} Q &= (Q^1, \ldots, Q^N)^\top = (X_1^1, X_2^1, X_3^1, \ldots, X_1^N, X_2^N, X_3^N)^\top \\ P &= (P^1, \ldots, P^N)^\top = (X_4^1, X_5^1, X_6^1, \ldots, X_4^N, X_5^N, X_6^N)^\top \\ \Gamma &= \mathrm{diag}[a_1, a_1, b_1, \ldots, a_N, a_N, b_N], \\ \Sigma &= \mathrm{diag}[\varepsilon_1, \sigma_1, \varepsilon_1, \ldots, \varepsilon_N, \sigma_N, \varepsilon_N] \end{aligned}$$

Simulation of stochastic Hamiltonian-type system

We can rewrite

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t),$$

as

$$dX(t) = (AX(t) + N(X(t))\,dt + \Sigma_0 dW(t),$$

with $X(t) = (Q(t), P(t))^T$ and

$$A = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad N(X(t)) = N(Q(t)) = \begin{pmatrix} \mathbb{O}_{3N} \\ G(Q(t)) \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}.$$

We will use *splitting schemes* ($\supseteq$ leap-frog) to simulate from it

Splitting integrators for the multi-population sJR-NMM

$$
\begin{aligned}
dX(t) &= [AX(t) + N(X(t))]dt + \Sigma_0 dW(t) \\
&= \left(\begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix} X(t) + \begin{pmatrix} \mathbb{O}_{3N} \\ G(Q(t)) \end{pmatrix} \right) dt + \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t).
\end{aligned}
$$

Step 1: Split the equation into explicitly solvable subequations.

Step 2: Derive the explicit solutions of the subequations.

Step 3: Compose the derived explicit solutions.

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\begin{aligned}
dX^{[1]}(t) &= AX^{[1]}(t)dt+\Sigma_0 dW(t) \\
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Step 2: Derive the explicit solutions of the subequations.

$$X^{[1]}(t_{i+1}) = \varphi_\Delta^{[1]}\left(X^{[1]}(t_i)\right) = e^{A\Delta}X^{[1]}(t_i)+\xi_i(\Delta),$$

with $\xi(\Delta) \sim N(0_{6N}, C(\Delta)), \mathrm{Cov}(\Delta) = \int_0^\Delta e^{A(\Delta-s)}\Sigma_0\Sigma_0^\top \left(e^{A(\Delta-s)}\right)^\top ds$ and

$$e^{F\Delta} = \begin{pmatrix} e^{-\Gamma\Delta}(\mathbb{I}_{3N}+\Gamma\Delta) & e^{-\Gamma\Delta}\Delta \\ -\Gamma^2 e^{-\Gamma\Delta}\Delta & e^{-\Gamma\Delta}(\mathbb{I}_{3N}-\Gamma\Delta) \end{pmatrix} =: \begin{pmatrix} \vartheta(\Delta) & \kappa(\Delta) \\ \vartheta'(\Delta) & \kappa'(\Delta) \end{pmatrix}$$

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$$\mathrm{Cov}(\Delta) = \begin{pmatrix} \frac{1}{4}\Gamma^{-3}\Sigma^2\left(\mathbb{I}_{3N}+\kappa(\Delta)\vartheta'(\Delta)-\vartheta^2(\Delta)\right) & \frac{1}{2}\Sigma^2\kappa^2(\Delta) \\ \frac{1}{2}\Sigma^2\kappa^2(\Delta) & \frac{1}{4}\Gamma^{-1}\Sigma^2\left(\mathbb{I}_{3N}+\kappa(\Delta)\vartheta'(\Delta)-\kappa'^2(\Delta)\right) \end{pmatrix}$$

Step 3: Compose the derived explicit solutions.

Splitting integrators for the multi-population sJR-NMM

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$$\begin{aligned} X^{[1]}(t_{i+1}) &= \varphi_\Delta^{[1]}\left(X^{[1]}(t_i)\right) = e^{A\Delta}X^{[1]}(t_i) + \xi_i(\Delta), \\ X^{[2]}(t_{i+1}) &= \varphi_\Delta^{[2]}\left(X^{[2]}(t_i)\right) = X^{[2]}(t_i) + \Delta \begin{pmatrix} 0_{3N} \\ G(Q^{[2]}(t_i)) \end{pmatrix}. \end{aligned}$$

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\end{aligned}
$$

Step 3: Compose the derived explicit solutions.

$$
\widetilde{X}^{\mathrm{S}}(t_{i+1}) = \left(\varphi_{\Delta/2}^{[2]} \circ \varphi_\Delta^{[1]} \circ \varphi_{\Delta/2}^{[2]}\right)\left(\widetilde{X}^{\mathrm{S}}(t_i)\right). \tag{2}
$$

Warwick Statistics

Splitting integrators for the multi-population sJR-NMM

Algorithm 1 Strang splitting scheme for the N-population stochastic JR-NMM

Input: Initial value X_0, step size Δ, number of time steps m in $[0, T]$ and model parameters

Output: Approximated path of $(X(t))_{t\in[0,T]}$ at discrete times $t_i = i\Delta$, $i = 0, \dots, m$, $t_m = T$.

Set $\widetilde{X}^{\mathrm{S}}(t_0) = X_0$

for $i = 0 : (m-1)$ **do**

    Set $X^{[2]} = \widetilde{X}^{\mathrm{S}}(t_i) + \frac{\Delta}{2}\begin{pmatrix} 0_{3N} \\ G(\widetilde{Q}^{\mathrm{S}}(t_i)) \end{pmatrix}$

    Set $X^{[1]} = e^{F\Delta} X^{[2]} + \xi_i(\Delta)$

    Set $\widetilde{X}^{S}(t_{i+1}) = X^{[1]} + \frac{\Delta}{2}\begin{pmatrix} 0_{3N} \\ G(Q^{[1]}) \end{pmatrix}$

end for

Return $\widetilde{X}^{\mathrm{S}}(t_i)$, $i = 0, \dots, m$.

Properties of the derived splitting scheme

The derived splitting scheme

1. is **mean-square convergent order 1** if $N(X(t))$ is globally Lipschitz (similar results for one-sided globally Lipschitz with polynomial growth[2].

2. is **1-step hypoelliptic**.

3. satisfies a **discrete Lyapunov condition** $\Rightarrow$ it is **geometrically ergodic**.

[2] Buckwar et al. Appl. Num. Math. 2022

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(More to be discussed):

- It could be used for simulating Langevin dynamics in HMC.
- Better than leap-frog/MALT.

[2]Buckwar et al. Appl. Num. Math. 2022

What about inference?

SMC-ABC

Algorithm 3 Sequential Monte Carlo ABC (SMC-ABC)

1: Set $t := 1$.
2: **for** $i = 1, \ldots, N$ **do**
3:     **repeat**
4:         Sample $\theta^* \sim \pi(\theta)$.
5:         Generate $z^i \sim p(z|\theta^*)$ from the model.
6:         Compute summary statistic $s^i = S(z^i)$.
7:     **until** $||s^i - s_y|| < \delta_1$
8:     Set $\theta_1^{(i)} := \theta^*$
9:     set $\tilde{w}_1^{(i)} := 1$.
10: **end for**
11: Obtain δ_2 and update the scaling factors for the summary statistics.
12: **for** $t = 2, \ldots, T$ **do**
13:     **for** $i = 1, \ldots, N$ **do**
14:         **repeat**
15:             Randomly pick (with replacement) θ^* from the weighted set $\{\theta_{t-1}^{(i)}, w_{t-1}^{(i)}\}_{i=1}^N$.
16:             Sample $\theta^{**} \sim q_t(\cdot|\theta^*)$.
17:             if $\pi(\theta^{**}) = 0$ go to step 16, otherwise continue.
18:             Generate $z^i \sim p(z|\theta^{**})$ from the model.
19:             Compute summary statistic $s^i = S(z^i)$.
20:         **until** $||s^i - s_y|| < \delta_t$
21:         Set $\theta_t^{(i)} := \theta^{**}$
22:         set $\tilde{w}_t^{(i)} = \pi(\theta_t^{(i)}) / \sum_{j=1}^N w_{t-1}^{(j)} q_t(\theta_t^{(i)}|\theta_{t-1}^{(j)})$.
23:     **end for**
24:     Normalise the weights: $w_t^{(i)} := \tilde{w}_t^{(i)} / \sum_{j=1}^N \tilde{w}_t^{(j)}$.
25:     Decrease the current δ and update the scaling factors for the summary statistics.
26: **end for**
27: *Output:*
28: A set of weighted parameter vectors $(\theta_T^{(1)}, \tilde{w}_T^{(1)}), \ldots, (\theta_T^{(N)}, \tilde{w}_T^{(N)}) \sim \pi_{\delta_T}(\theta|s_y)$.

Adjusted SMC-ABC

for $i = 1 : M$ **do**
  repeat
    Randomly pick (with replacement) θ_c from the weighted set $\{\Theta_{c,t-1}, w_{t-1}\}$
    Perturb θ_c to obtain θ_c^* from $q_t^c(\cdot|\theta_c)$.
    Sample θ_d^k, $k = 1, \ldots, d_n$, from $\mathrm{Bernoulli}(\hat{p}_t^k)$, where $\hat{p}_t^k = \frac{1}{M} \sum_{l=1}^{M} \theta_{d,t-1}^{k,(l)}$.
    Perturb $\theta_d = (\theta_d^1, \ldots, \theta_d^{d_n})$ to obtain θ_d^* from $q_t^d(\cdot|\theta_d)$.
    Conditioned on $\theta^* = (\theta_c^*, \theta_d^*)$, simulate a dataset $\tilde{y}_{\theta^*}$ from the model.
    Compute the summaries $s(\tilde{y}_{\theta^*})$.
    Calculate the distance $D = d\big(s(y), s(\tilde{y}_{\theta^*})\big)$.
  until $D < \delta_t$
  Set $\theta_{d,t}^{(i)} = \theta_d^*$ and $\theta_{c,t}^{(i)} = \theta_c^*$
  Set $\tilde{w}_t^{(i)} = \pi^c\left(\theta_{c,t}^{(i)}\right) / \sum_{l=1}^{M} w_{t-1}^{(l)} \mathscr{K}_t^c\left(\theta_{c,t}^{(i)} \middle| \theta_{c,t-1}^{(l)}\right)$
end for
Normalise the weights $w_t^{(i)} = \tilde{w}_t^{(i)} / \sum_{l=1}^{M} \tilde{w}_t^{(i)}$, for $j = 1, \ldots, M$

Choice of perturbation kernels

q^c_θ: Optimised Gaussian kernels as in Filippi et al. 2013 (alternatively: copula-based samplers, Picchini and Tamborrino, 2022).

Discrete kernel: a value $\theta^d_k, k = 1, \ldots, d_n$, sampled from a Bernoulli distribution at iteration t is either kept with (fixed) probability q_{stay} or perturbed to $1 - \theta^d_k$, i.e.

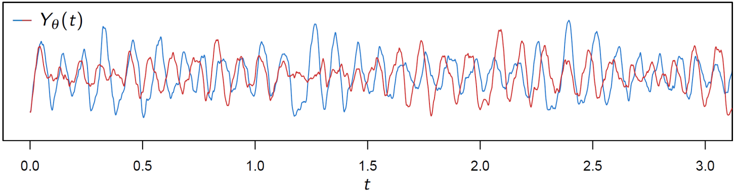
$$q^d_t\left(\theta^{(i)}_{d,t}\middle|\theta^{(l)}_{d,t-1}\right) = \prod_{k=1}^{d_n} q^{d,k}_t\left(\theta^{k,(i)}_{d,t}\middle|\theta^{k,(l)}_{d,t-1}\right) = \prod_{k=1}^{d_n}\left(p^{k,(l)}_t\right)^{\theta^{k,(i)}_{d,t}}\left(1 - p^{k,(l)}_t\right)^{1-\theta^{k,(i)}_{d,t}},$$

where

$$p^{k,(l)}_t = \begin{cases} 1 - q_{\text{stay}}, & \text{if } \theta^{k,(l)}_{d,t-1} = 0\,. \end{cases}$$

Choice of Summary Statistics

Accept θ^* if $d(s(y), s(\tilde{y}_{\theta^*})) < \delta_t$.



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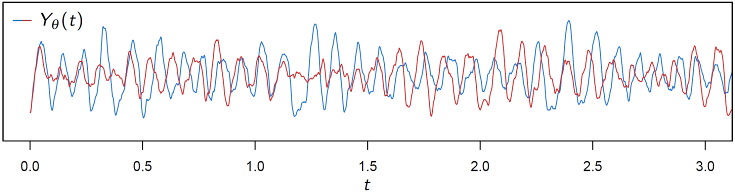
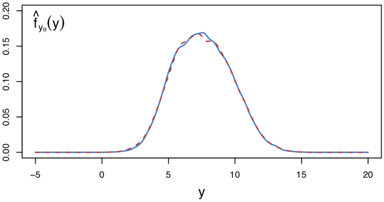
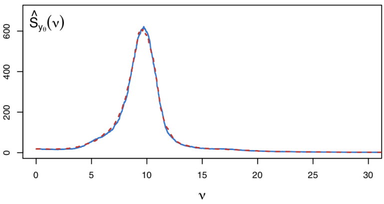


$\Longrightarrow$ Derive summaries based on the characterising model properties: map the data into something fully characterised by θ.





Choice of Summary Statistics

$$s(y) := \left\{f_k, S_k, Z_{jk}, R_{jk}\right\}_{j,k=1,\ldots,N, j\neq k}.$$

* f_k: invariant density of Y^k.

* **Spectral density** S_k of Y^k:

$$S_k(\nu) = \mathscr{F}\{R_k\}(\nu) = \int_{-\infty}^{\infty} R_k(\tau) e^{-i2\pi\nu\tau}\, d\tau, \quad k \in \{1, \ldots, N\},$$

where ν denotes the frequency and $R_k(\tau) = \mathbb{E}[Y^k(t) Y^k(t+\tau)], \quad k \in \{1, \ldots, N\}$.

* **Cross-spectral density** S_{jk} of Y^j and Y^k:

$$S_{jk}(\nu) = \mathscr{F}\{R_{jk}\}(\nu) = \int_{-\infty}^{\infty} R_{jk}(\tau) e^{-i2\pi\nu\tau}\, d\tau,$$

where $R_{jk}(\tau) = \mathbb{E}[Y^j(t) Y^k(t+\tau)], \quad j,k \in \{1, \ldots, N\},\ j \neq k$.

* **Magnitude Square Coherence (MSC)**:

$$Z_{jk}(\nu) := \frac{|S_{jk}(\nu)|^2}{S_j(\nu) S_k(\nu)}, \quad j,k \in \{1, \ldots, N\},\ j \neq k,$$

where $|\cdot|$ denotes the magnitude.

Choice of distance measure

We use the Integrate Absolute Error (IAE)[3]

$$\mathrm{IAE}(g_1, g_2) := \int_{\mathbb{R}} |g_1(x) - g_2(x)| \; dx \;\in\; \mathbb{R}^+,$$

to compute

$$D(s(y), s(\tilde{y}_\theta)) \quad := \quad v_1 \frac{1}{N} \sum_{k=1}^{N} \mathrm{IAE}(\hat{S}_k, \tilde{S}_k) + v_2 \frac{1}{N(N-1)/2} \sum_{j=1, k>j}^{N} \mathrm{IAE}(\hat{Z}_{jk}, \tilde{Z}_{jk})$$
$$+ v_3 \frac{1}{N(N-1)} \sum_{j,k=1, j\neq k}^{N} \mathrm{IAE}(\hat{R}_{jk}, \tilde{R}_{jk}) + v_4 \frac{1}{N} \sum_{k=1}^{N} \mathrm{IAE}(\hat{f}_k, \tilde{f}_k),$$

The weights $v_l \geq 0$, $l = 1, 2, 3, 4$, are chosen such that the different summary functions have a comparable impact on the distance measure.

[3] Buckwar, Tamborrino, Tubikanec, Stat. Comput. 2020

Parameters of interest

$(N+2+N(N-1))$-dimensional parameter vector

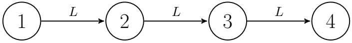
$$\theta = (\underbrace{A_1, \ldots, A_N, L, c}_{\theta_c}, \underbrace{\text{vec}(\mathscr{P})}_{\theta_d}),$$

with

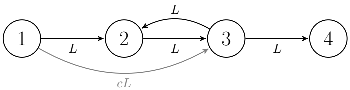
- A_k: Average excitatory synaptic gains.
- $\mathscr{P}$: directed connectivity parameters $\theta_d = \mathscr{P} = (\rho_{jk})_{j,k=1,\ldots,N}$, with $\rho_{ji} = \{0,1\}$.
- (L, c) entering into the coupling parameters K_{jk} as

$$K_{jk} := c^{|j-k|-1} L,$$

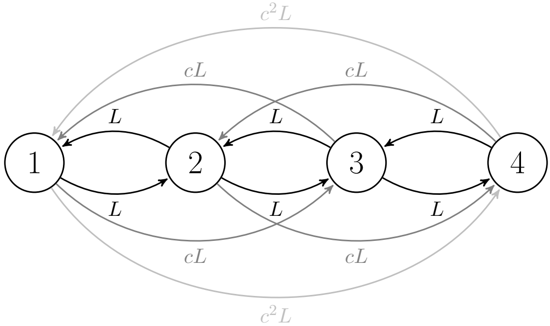
- $L > 0$: coupling strength parameter
- $0 \ll c < 1$ determines how fast the the network coupling strength decreases with distance.



(a)

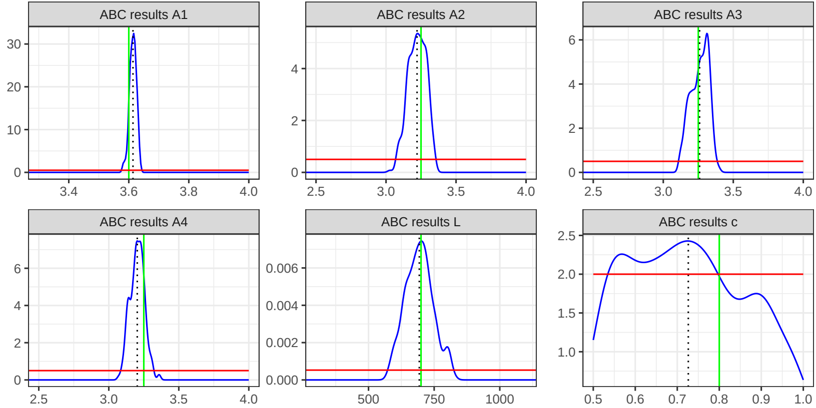


(b)

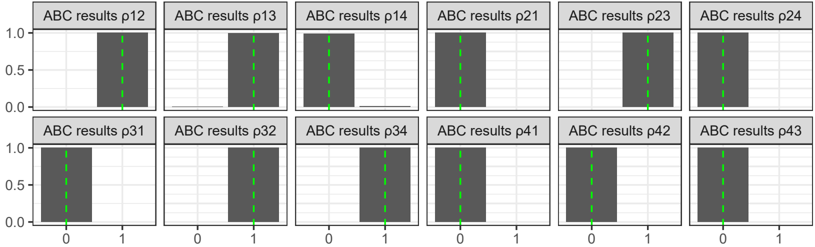


(c)

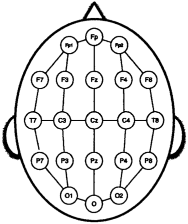
Partially connected network



Partially connected network



Back to real data

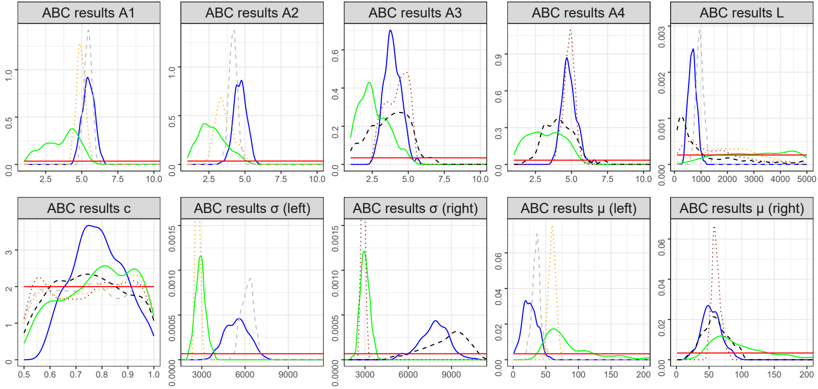


$$K = \begin{pmatrix} - & K_{12} & K_{13} & K_{14} \\ K_{21} & - & K_{23} & K_{24} \\ K_{31} & K_{32} & - & K_{34} \\ K_{41} & K_{42} & K_{43} & - \end{pmatrix} = \begin{pmatrix} - & L & c^2L & c^3L \\ L & - & cL & c^2L \\ c^2L & cL & - & L \\ c^3L & c^2L & L & - \end{pmatrix},$$

* b and C chosen from pilot study, other quantities fixed according to standard values.

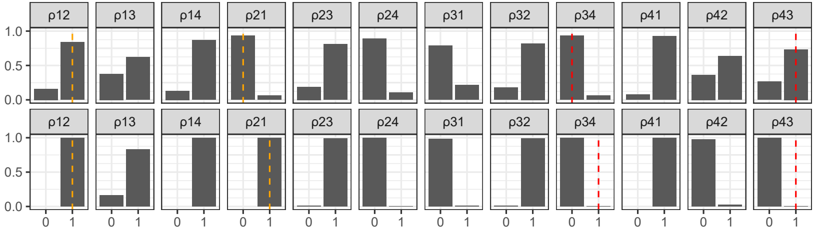
Parameter of interest: (10+12)-dimensional

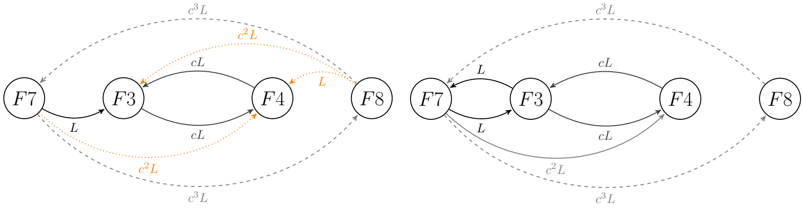
$$\theta = (A_1, A_2, A_3, A_4, L, c, \sigma_l, \sigma_r, \mu_l, \mu_r, \mathrm{vec}(\mathscr{P})).$$

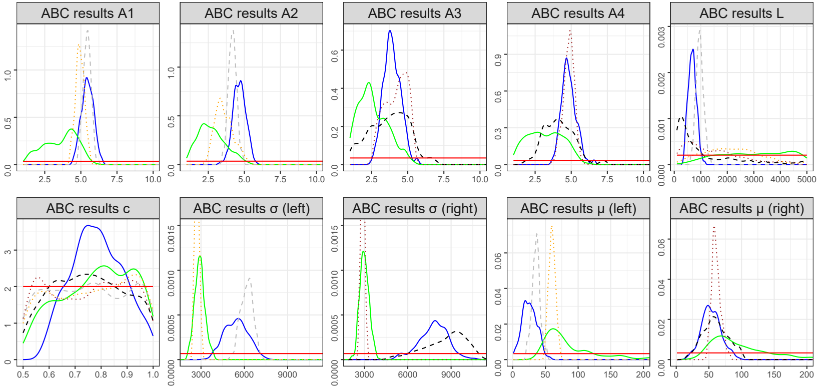


Before seizure: solid green ($N = 4$)

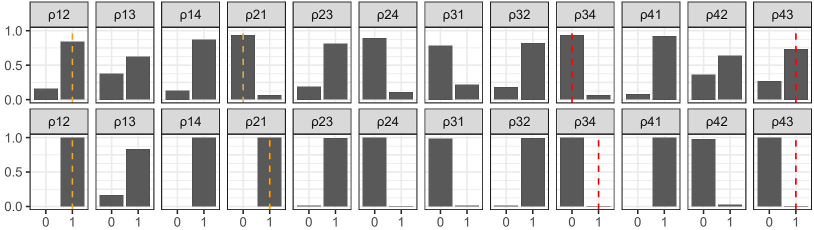
During seizure: solid blue ($N = 4$)

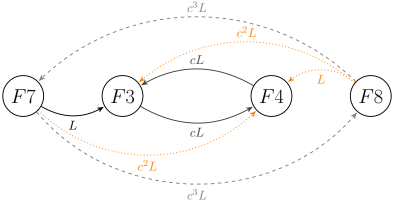

ρ12
ρ13
ρ14
ρ21
ρ23
ρ24
ρ31
ρ32
ρ34
ρ41
ρ42
ρ43
1.0
0.5
0.0
0
1

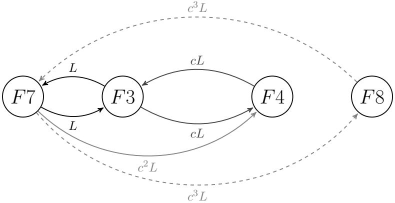

$F7$
$F3$
$F4$
$F8$
L
cL
c^2L
c^3L



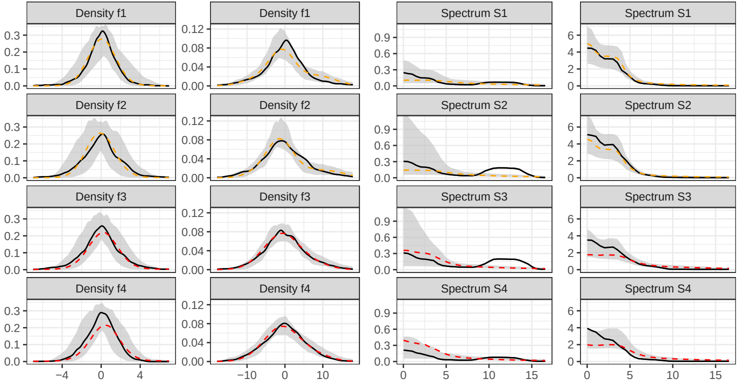
Before seizure: solid green ($N = 4$), dotted orange ($N = 2$, LH), dotted brown ($N = 2$, RH).
During seizure: solid blue ($N = 4$), dashed grey ($N = 2$, LH), dashed black ($N = 2$, RH).


ρ12 ρ13 ρ14 ρ21 ρ23 ρ24 ρ31 ρ32 ρ34 ρ41 ρ42 ρ43
1.0
0.5
0.0
0 1


F7
F3
F4
F8
L
cL
c²L
c³L


F7
F3
F4
F8
L
cL
c²L
c³L

Fitted summaries



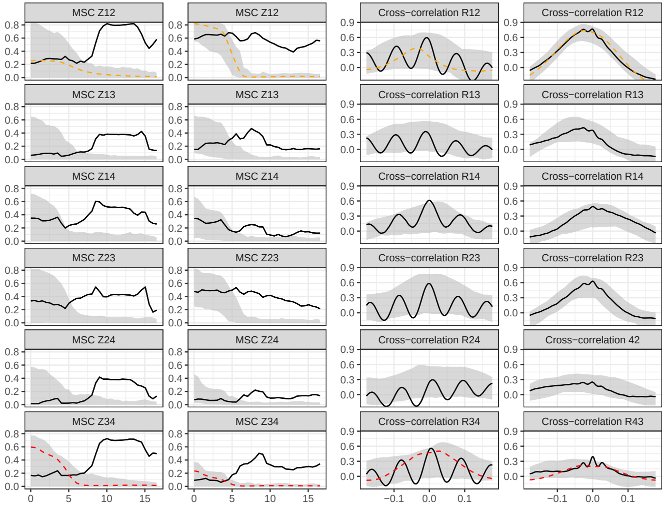
Odd panels: before seizure. Even panels: during seizure.

Solid black lines: Summaries derived from the EEG datasets.

Grey areas: Range of the summaries obtained from synthetic datasets simulated using the kept posterior samples from the full model.

Warwick Statistics

Fitted summaries



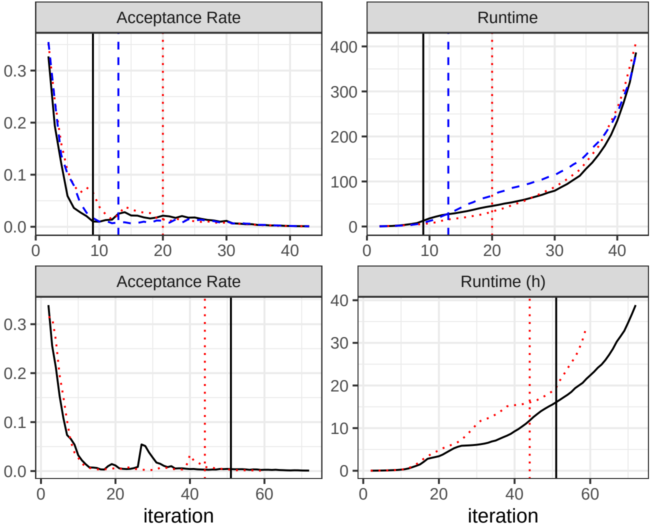
Some references

Today Ditlevsen, Tamborrino, Tubikanec.
Network inference in a stochastic multi-population neural mass model via approximate Bayesian computation.
Preprint at arXiv:2306.15787, 2023.

- Buckwar, Tamborrino, Tubikanec.
  Spectral density-based and measure-preserving ABC for partially observed diffusion processes. An illustration on Hamiltonian SDEs.
  Stat. Comput., 30, 627–648, 2020.
- Picchini, Tamborrino.
  Guided sequential ABC for intractable Bayesian models.
  Preprint at arXiv:2206.12235, 2022.
- Buckwar, Samson, Tamborrino, Tubikanec.
  A splitting method for SDEs with locally Lipschitz drift. An illustration on the FitzHugh-Nagumo model.
  App. Num. Math. 179, 191–220, 2022.

Some interesting ongoing/forthcoming activities

- OneWorldABC (every last Thursday of the month)
  www.warwick.ac.uk/oneworldabc
- BioInference2024, 5th-7th June 2024, Warwick.
  https://bioinference.github.io/2024/



Acceptance Rate
Runtime
Acceptance Rate
Runtime (h)
0.3
0.2
0.1
0.0
400
300
200
100
0
0
10
20
30
40
0
20
40
60
iteration
iteration

Specific choices

- M=500.
- $T = 20, \Delta_{\text{sim}} = 10^{-4}, \Delta_{\text{obs}} = 210^{-3} \Rightarrow n = 10^4$.
- δ_1 obtained via a reference table acceptance-rejection ABC pilot run. Under $\pi(\theta)$, we produce 10^4 distances and then choose $\delta_1 = \text{median}(D_1, \dots, D_{10^4})$.
- $\delta_t = \text{percentile}(D_1^{(t-1)}, \dots, D_M^{(t-1)})$, with percentile $= 50\%$ if accept. rate $> 1\%, 75\%$ otherwise.
- Stopping criterion: acceptance rate below 0.1%.

Table 1: Standard parameter values for the Jansen and Rit Neural Mass Model [1, 18, 42].

Parameter	Meaning	Standard value
A	Average excitatory synaptic gain	3.25 mV
B	Average inhibitory synaptic gain	22 mV
a	Membrane time constant of excitatory postsynaptic potential	100 s^{-1}
b	Membrane time constant of inhibitory postsynaptic potential	50 s^{-1}
C	Average number of synapses between the subpopulations	135
C_1, C_2	Avg. no. of synaptic contacts in the excitatory feedback loop	C, 0.8 C
C_3, C_4	Avg. no. of synaptic contacts in the inhibitory feedback loop	0.25 C, 0.25 C
$\nu_{\max}$	Maximum firing rate (Maximum of the sigmoid function)	5 s^{-1}
v_0	Value for which 50% of the maximum firing rate is attained	6 mV
γ	Determines the slope of the sigmoid function at v_0	0.56 mV^{-1}

Algorithm 2 Adjusted SMC-ABC for network inference (nSMC-ABC)

Input: Summaries $s(y)$ of the observed data y, prior distributions π^c and π^d, perturbation kernels $\mathcal{K}_r^c$ and $\mathcal{K}_r^d$, number of kept samples per iteration M, initial threshold δ_1

Output: Samples from the nSMC-ABC posterior

Set $r = 1$
for $j = 1 : M$ **do**
    repeat
        Sample θ_d from π^d and θ_c from π^c, and set $\theta = (\theta_c, \theta_d)$
        Conditioned on θ, simulate a synthetic dataset $\tilde{y}_\theta$ from the observed output Y
        Compute the summaries $s(\tilde{y}_\theta)$
        Calculate the distance $D = d\big(s(y), s(\tilde{y}_\theta)\big)$
    until $D < \delta_1$
    Set $\theta_{d,1}^{(j)} = \theta_d$ and $\theta_{c,1}^{(j)} = \theta_c$
end for
Initialize the weights by setting each entry of $w_1 = (w_1^{(1)}, \ldots, w_1^{(M)})$ to $1/M$
repeat
    Set $r = r + 1$
    Determine $\delta_r < \delta_{r-1}$
    for $j = 1 : M$ **do**
        repeat
            Sample θ_c from the weighted set $\{\Theta_{c,r-1}, w_{r-1}\}$
            Perturb θ_c to obtain θ_c^* from $\mathcal{K}_r^c(\cdot|\theta_c)$
            Sample θ_d^k, $k = 1, \ldots, d_n$, from Bernoulli$(\hat{p}_r^k)$, where $\hat{p}_r^k = \frac{1}{M}\sum_{l=1}^{M} \theta_{d,r-1}^{k,(l)}$
            Perturb $\theta_d = (\theta_d^1, \ldots, \theta_d^{d_n})$ to obtain θ_d^* from $\mathcal{K}_r^d(\cdot|\theta_d)$
            Conditioned on $\theta^* = (\theta_c^*, \theta_d^*)$, simulate a dataset $\tilde{y}_{\theta^*}$ from the observed output Y
            Compute the summaries $s(\tilde{y}_{\theta^*})$
            Calculate the distance $D = d\big(s(y), s(\tilde{y}_{\theta^*})\big)$
        until $D < \delta_r$
        Set $\theta_{d,r}^{(j)} = \theta_d^*$ and $\theta_{c,r}^{(j)} = \theta_c^*$
        Set $\tilde{w}_r^{(j)} = \pi^c\left(\theta_{c,r}^{(j)}\right) / \sum_{l=1}^{M} w_{r-1}^{(l)} \mathcal{K}_r^c\left(\theta_{c,r}^{(j)} \middle| \theta_{c,r-1}^{(l)}\right)$
    end for
    Normalise the weights $w_r^{(j)} = \tilde{w}_r^{(j)} / \sum_{l=1}^{M} \tilde{w}_r^{(l)}$, for $j = 1, \ldots, M$
until stopping criterion is reached