Tilting the Classroom: Engaging Students in Large Lectures

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Transition to university
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- The arrangements change.
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• For most of us, there isn’t an imminent dramatic change in teaching arrangements.
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• The arrangements change.
• The mathematics changes.
• This is going to remain true for the foreseeable future.
• For most of us, there isn’t an imminent dramatic change in teaching arrangements.
• So what can we do to help our students and make our own lives easier?
Teaching undergraduates
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• I’m not going to tell you how to teach because I don’t know that, and I think it’s in large part a personal thing.
Teaching undergraduates

• I’m not going to tell you how to teach because I don’t know that, and I think it’s in large part a personal thing.

• I’m going to tell you some things about how I do it.
Teaching undergraduates

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• 3 overarching principles.
Teaching undergraduates

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• 18 practices with lots of examples.
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• These practices work well together but also separately:
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  • Over the years I’ve gradually added some and extended my use of others;
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• 3 overarching principles.

• 18 practices with lots of examples.

• These practices work well together but also separately:
  • Over the years I’ve gradually added some and extended my use of others;
  • It’s perfectly practical to start with any one of them.
Why that’s important
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• Which symbol goes in the gap?
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⇒, ↔, ↔, none of these
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⇒, ⇐, ⊳, none of these

innovative teaching       good teaching
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innovative teaching      good teaching
Why that’s important

• Which symbol goes in the gap?

    \(\Rightarrow, \leftrightarrow, \equiv, \textcolor{green}{\text{none of these}}\)

    innovative teaching      good teaching

• I find the innovation agenda deeply troubling - I think it demoralises and puts off people who are keen to improve.
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• I believe that incremental change and improvement across the board is how we’ll really get better learning and more student enjoyment of our subjects.
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• I find the innovation agenda deeply troubling - I think it demoralises and puts off people who are keen to improve.

• I believe that incremental change and improvement across the board is how we’ll really get better learning and more student enjoyment of our subjects.

• (I think that we’ll then get better NSS scores etc., but that will be a consequence - it should not be an aim in itself.)
Claims and disclaimers
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  • Its fundamental definitions are logically complex;
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  - It’s a theoretical, theorems-and-proofs module, completely different from procedure-based mathematics.
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• I don’t know how to make every student understand it deeply and perform well in the exam.
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• It’s difficult, mostly for two reasons:
  • Its fundamental definitions are logically complex;
  • It’s a theoretical, theorems-and-proofs module, completely different from procedure-based mathematics.
• I don’t know how to make every student understand it deeply and perform well in the exam.
• I do know how to help many students engage with the complex ideas and recognise their own growth.
Principles
Principle 1
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There is no point in the lecturer covering the material if the students don’t.
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- In about 50% of my own undergraduate lectures I understood nothing.
- I’m not an idiot: I know that I have obligations to teach what’s on the curriculum.
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• In about 50% of my own undergraduate lectures I understood nothing.

• I’m not an idiot: I know that I have obligations to teach what’s on the curriculum.

• I’m not a fantasist: I know that I have only partial control over what’s learned, and that students won’t understand everything in every lecture.
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• I’m not an idiot: I know that I have obligations to teach what’s on the curriculum.

• I’m not a fantasist: I know that I have only partial control over what’s learned, and that students won’t understand everything in every lecture.

• But it really helps if students are given good opportunities to engage and to re-engage.
Principle 2
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Students are not lazy or bad people.
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• They just have moments of weakness and respond poorly to sensations of failure, exactly like the rest of us.
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• They just have moments of weakness and respond poorly to sensations of failure, exactly like the rest of us.

• They need to learn not just content but organisational skills and resilience - I can help with that.
Principle 3
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Learning is about what the students do more than it is about what the lecturer does.
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• Of course, the lecturer guides and orchestrates student activity.

• But when I’m planning I think less and less about what I do in the lecture theatre, and more and more about what students do both inside it and in their independent study time.
Practices: Organisation
Organisation
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• I want all the intellectual energy available for focusing on the mathematics.
Organisation

- I want all the intellectual energy available for focusing on the mathematics.
- I want all the emotional energy available for being resilient in the face of struggle and tiredness.
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- I want all the emotional energy available for being resilient in the face of struggle and tiredness.
- Four practices: announcements, break, notes, routine.
Organisation: announcements
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• I put (prepared) handwritten announcements on the visualiser as students come in.
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• They say things like:
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- They say things like:

  Good morning.
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  Please pick up a set of notes from the front.
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  Remember that on Monday at 11 we have our first test; the B questions appear at the end of this week’s problems.
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  Tutors will mark and return tests for the first years, I will mark those for the second years and they’ll be returned via the office.
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  Turn to page 54 - what is your answer to this morning’s question?
Organisation: break
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- 2-minute break in the middle of every lecture.
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• Between about 22 and 30 minutes past, depending on content.
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Organisation: notes
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- Gappy notes distributed weekly.
Organisation: notes

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- Students have copies.
Organisation: notes

• Gappy notes distributed weekly.
• Students have copies.
• So do I.
Organisation: notes

• Gappy notes distributed weekly.
• Students have copies.
• So do I.
• ~4 pages per lecture.
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- Problem sheet attached to back.
Organisation: notes

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- Page numbers contiguous.
Organisation: routine
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• (Monday 11am, Monday 5pm, Wednesday 11am.)
Organisation: routine

- (Monday 11am, Monday 5pm, Wednesday 11am.)
- Monday morning: pick up notes on the way in.
Organisation: routine

- (Monday 11am, Monday 5pm, Wednesday 11am.)
- Monday morning: pick up notes on the way in.
- Between Monday and Wednesday: reading a few pages printed in notes.
Organisation: routine

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- Monday morning: pick up notes on the way in.
- Between Monday and Wednesday: reading a few pages printed in notes.
- Wednesday: start with 10 true/false questions.
Organisation: routine

• (Monday 11am, Monday 5pm, Wednesday 11am.)
• Monday morning: pick up notes on the way in.
• Between Monday and Wednesday: reading a few pages printed in notes.
• Wednesday: start with 10 true/false questions.
• Completed notes on VLE on Wednesday.
Organisation: routine

• (Monday 11am, Monday 5pm, Wednesday 11am.)
• Monday morning: pick up notes on the way in.
• Between Monday and Wednesday: reading a few pages printed in notes.
• Wednesday: start with 10 true/false questions.
• Completed notes on VLE on Wednesday.
• Problem solutions on VLE on Friday.
Student feedback
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• There’s a clear routine & structure to the module and the lectures, it’s easy to know what’s going on.
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• I like how clear and organised it is about what we need to do, when we will be able to get solutions etc.
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• Short breaks in lectures so you can concentrate better.
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• Breaks in the middle of lectures to catch up and just evaluate everything so far.
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• Breaks in the middle of lectures to catch up and just evaluate everything so far.

• (Mini) breaks to have a “breather” or re-read pages.
Practices:
Study Guidance
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• I take less and less for granted in terms of what students know about how to study (Bjork, Dunlosky & Kornell, 2013).
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- Some first years know very little.
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- I offer practical information with a bit of encouragement.
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• Some have never had to study much outside of class in order to keep up.

• I offer practical information with a bit of encouragement.

• Three practices: clarify expectations, provide self-explanation training, give early feedback opportunity.
Study guidance: expectations
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- Week 2 reading (6 pages).
Study guidance: expectations

- Week 2 reading (6 pages).
- What Analysis is like.
Study guidance: expectations

• Week 2 reading (6 pages).
• What Analysis is like.
• Keeping up.
Study guidance: expectations

- Week 2 reading (6 pages).
- What Analysis is like.
- Keeping up.
- What to do (how much time to spend reading notes, trying problems).
Study guidance: expectations

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• What to do when you’re stuck (things to try, where and how to seek help).
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Here is what happens when I teach Analysis. In week 1, everyone is in a good mood because they’re starting something new. In weeks 2 and 3, there is a buildup of increasingly challenging material. In week 4, the mood in the lecture theatre is dreadful. The whole class has realized that this is difficult stuff and that it isn’t going to get any easier. Everyone hates Analysis and, by extension, quite a few people hate me. I am not fazed by this, though, because I have taught Analysis about twenty times now and I know what will happen next. In week 5, everyone will feel slightly better, even if no-one can quite explain why. In week 7, a small number of people will approach me and tell me shyly that, although Analysis is challenging, they’re starting to think they might like it. By the end of the course, these people will be telling anyone who will listen that Analysis is brilliant, and lots of other students will admit that now that they’re getting the hang of it, they can see why people think it’s a great subject.

The question for a new student, then, is how to handle it when the work gets difficult and you start to feel negative. Some students turn the negativity inwards: they lose confidence, experience self-doubt about their mathematical ability (‘Perhaps I’m not good enough for this?’), and sometimes become withdrawn. Others turn it outwards, expressing frustration and anger about their lecturers (‘He’s a terrible teacher!’) and sometimes, a bit nonsensically, about the mathematics itself (‘I don’t know why they’re teaching us this rubbish—this isn’t maths!’). These reactions both arise naturally when people feel a loss of control and consequently get defensive. But neither is very productive. So what is the alternative?
In Analysis, as in any undergraduate mathematics module, the big challenge is keeping up. If you’re taking a decent degree course, this will be difficult. No-one is trying to teach you stuff that you will find easy—what would be the point of that? Also, you will be busy, with other courses and with the rest of your life. So it is very unlikely that you will be on top of everything all the time. You should try not to be distressed by this, because distress doesn’t help—negative emotions just impede effective study. The thing to do is to accept that you will not always have perfect knowledge of everything, and work in an intelligent way that allows you to maintain *sufficient* knowledge of the *important* things.

When I say *sufficient* knowledge, I mean enough knowledge to give you a fighting chance of making sense of new material. By the time you are a few weeks into a module, you are unlikely to understand everything in every lecture—I certainly didn’t when I was an undergraduate. But you want to have enough under your belt that you can follow the big sweep of the theory development and understand some of the details. When I say the *important* things, I mean the central concepts that come up again and again. At any given time, it is unlikely that you will be able to explain the nuances of every proof, but you want to know the main definitions and theorems so that you can recognize when and how they are used in new work. With that in mind, here is what I would prioritize.

First, you absolutely must know your definitions. In Analysis, it is sometimes tempting to be lax about this, because many of the words used (‘increasing’, ‘convergent’, ‘limit’ etc.) have everyday meanings, and because concepts in Analysis can often be represented using diagrams. Both of these things will tempt you into thinking that intuitive understanding is sufficient. *It isn’t.* Definitions are central to any theory in advanced mathematics—
Study guidance: self-explanation training
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- Research-based training.
Study guidance: self-explanation training

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- Relate each line you read to earlier material and prior knowledge.
Study guidance: self-explanation training

- Research-based training.
- Relate each line you read to earlier material and prior knowledge.
- Differentiate self-explanation from monitoring and paraphrasing.
Study guidance: self-explanation training

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• Practice.

Self-Explanation Training for Mathematics Students
Study guidance: self-explanation training

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• Practice.
• Online at www.setmath.lboro.ac.uk.
Study guidance:
early feedback opportunity
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- Two years ago I finally did the thing that lecturer training courses and books say you should do.
Study guidance: early feedback opportunity

• Two years ago I finally did the thing that lecturer training courses and books say you should do.

• I gave out post-its with instruction to write (on different sides):
Study guidance: early feedback opportunity

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• I gave out post-its with instruction to write (on different sides):
  • Something that you like about the module or that’s going well;
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  • Something that you don’t like or that you’re concerned about or that you didn’t understand.
Study guidance: early feedback opportunity

- Two years ago I finally did the thing that lecturer training courses and books say you should do.
- I gave out post-its with instruction to write (on different sides):
  - Something that you like about the module or that’s going well;
  - Something that you don’t like or that you’re concerned about or that you didn’t understand.
- I went through these and put up lists of both.
THINGS PEOPLE LIKE

- T/F quizzes and recap tests (loads of people said this)
- Interactive lectures and the mix of activities (and this)
- Structure and provision of gappy notes (and this)
- Me being engaging / enthusiastic / positive / not too serious
- Learning new and challenging material
- Clear explanations (including diagrams / gestures etc.)
- Learning about logical reasoning / theory development
- Readings (for preparation and study skills)
THINGS PEOPLE ARE CONCERNED ABOUT

- Analysis is difficult
- The pace is too fast
- They're worried about constructing proofs
- My writing and/or talking is too fast
- A few people don't like the interactive class discussions
- Worried about falling behind / remembering everything
- Want more help or feedback outside class, but...
- ... worried about asking about simple things.
- Would like printed definitions sheet
- Would like Review used
- Would have liked a slower start
- First test too early
- Getting to grips with symbols
- Would like more worked examples
- Time for reading too short (and self-teaching demanding)
- Clashes / test timing
Activities
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• I provide opportunities to be wrong, to be right, and to recognise that you don’t know.
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• All are important for engagement and a sense of progress.
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• Handled well, a large class is the ideal environment for generating challenge, emotional investment and support.
Activities

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• All are important for engagement and a sense of progress.

• Handled well, a large class is the ideal environment for generating challenge, emotional investment and support.

• Basically, I aim to orchestrate emotional responses: instead of an unmemorable “yeah okay, yeah okay”, I want the thinking to go “oh I know that…oh no wait, maybe I don’t…oh, that looked easy but it isn’t…haha! yeah! I get it now!”
Activities
Activities

• Gappy notes are great for this.
Activities

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• They mean that some important stuff that you want to say but not spend time writing can be pre-printed.
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• They mean that activities can be converted so that instead of listening (and perhaps copying) they require higher-level thinking (and probably communication).
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- They mean that activities can be converted so that instead of listening (and perhaps copying) they require higher-level thinking (and probably communication).
- They facilitate variety and short-and-snappy tasks.
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- They mean that activities can be converted so that instead of listening (and perhaps copying) they require higher-level thinking (and probably communication).
- They facilitate variety and short-and-snappy tasks.
- Five practices: filling things in, deciding, reading and explaining, reviewing via true/false questions, tests.
Activities: filling things in
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• If something can be filled in by the students without my assistance, it probably should be.
Activities: filling things in

• If something can be filled in by the students without my assistance, it probably should be.

• This works well with routine extensions, applications to examples, and more conceptual thought about mathematical claims.
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• Students can work on these together.
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- Responses can sometimes be gathered by collective shouting out (a throwing gesture works well for this).
Activities: filling things in

- If something can be filled in by the students without my assistance, it probably should be.
- This works well with routine extensions, applications to examples, and more conceptual thought about mathematical claims.
- Students can work on these together.
- Responses can sometimes be gathered by collective shouting out (a throwing gesture works well for this).
- (Note: I don’t have them do routine calculations, usually - those take too long and can be done at home.)
Theorem

\[ \lim_{n \to \infty} x^n = \begin{cases} \infty & \text{if} \\ 1 & \text{if} \\ 0 & \text{if} \end{cases} \quad (x^n) \text{ has no limit otherwise.} \]
**Theorem**

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\infty & \text{if} \\
1 & \text{if} \\
0 & \text{if}
\end{cases} \quad (x^n) \text{ has no limit otherwise.}$$

Here are some more definitions. Fill in those that are not complete.

**Definition:** A sequence \((a_n)\) is *bounded above* if and only if \(\exists u \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}, a_n \leq u.\)

**Definition:** \(u \in \mathbb{R}\) is an *upper bound* for \((a_n)\) if and only if \(\forall n \in \mathbb{N}, a_n \leq u.\)

**Definition:** A sequence \((a_n)\) is *bounded below*

**Definition:** \(l \in \mathbb{R}\) is an *lower bound* for \((a_n)\)
Definition

Let $A$ be a nonempty subset of $\mathbb{R}$. Then $L \in \mathbb{R}$ is the \textit{infimum} of $A$ if and only if:

1. 

2.

What do you think the infimum is sometimes called?

<table>
<thead>
<tr>
<th>set</th>
<th>bounded above?</th>
<th>three upper bounds</th>
<th>supremum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, 1]$</td>
<td></td>
<td></td>
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<tr>
<td>$(0, 1)$</td>
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<tr>
<td>$[0, \infty)$</td>
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<td>${x \in \mathbb{R} \mid x^2 &lt; 2}$</td>
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3.3.3 Axioms for the real numbers

1. \( \forall x, y \in \mathbb{R}, x + y \in \mathbb{R}. \)
2. \( \forall x, y \in \mathbb{R}, xy \in \mathbb{R}. \)
3. \( \forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z). \)
4. \( \forall x, y \in \mathbb{R}, x + y = y + x. \)
5. \( \exists 0 \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, x + 0 = x = 0 + x. \)
6. \( \forall x \in \mathbb{R} \exists (-x) \in \mathbb{R} \text{ s.t. } x + (-x) = 0 = (-x) + x. \)
7. \( \forall x, y, z \in \mathbb{R}, (xy)z = x(yz). \)
8. \( \forall x, y \in \mathbb{R}, xy = yx. \)
9. \( \exists 1 \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, x \cdot 1 = x = 1 \cdot x. \)
10. \( \forall x \in \mathbb{R} \exists x^{-1} \in \mathbb{R} \text{ s.t. } xx^{-1} = 1 = x^{-1}x. \)
11. \( \forall x, y, z \in \mathbb{R}, x(y + z) = xy + xz. \)
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Here is a list of names of these axioms (they don’t all have names). Match them up.

- closure under multiplication
- associativity of multiplication
- existence of a multiplicative identity
- trichotomy
- associativity of addition
- distributivity of multiplication over addition
- existence of multiplicative inverses
- commutativity of addition
- commutativity of multiplication
- closure under addition
- transitivity
- existence of an additive identity
- existence of additive inverses
Activities: deciding
Activities: deciding

• Discuss for 30 seconds / one minute / three minutes...
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- My favourites are the questions that I know will split the class 50/50.
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- My favourites are the questions that I know will split the class 50/50.

- I sometimes say “Now explain why the right answer is right.”
What symbol goes in the gap in this theorem? \( \Rightarrow, \leftrightarrow \text{ or } \Leftrightarrow? \)

Theorem

For \( a \neq 0 \), \((a_n) \to a\) \quad \(|a_n| \) \to |a|.

Theorem

\((a_n)\) is convergent \quad \((a_n)\) is bounded.

Theorem (null sequence test)

\[ \sum_{n=1}^{\infty} a_n \text{ is convergent} \quad \text{if} \quad (a_n) \to 0. \]
True or false?

If a set $A \subseteq \mathbb{R}$ has supremum $U$, then $U \in A$.

If a set $A \subseteq \mathbb{R}$ has supremum $\sup A$ and if we define $-A = \{-a \mid a \in A\}$, then $\sup(-A) = -\sup A$.

If a set $A \subseteq \mathbb{R}$ has supremum $\sup A$, then $\forall \varepsilon > 0 \ \exists a \in A$ such that $\sup A - \varepsilon < a \leq \sup A$. 
Activities: read and explain
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• Read this definition or theorem and explain to the person next to you what it means (and why it’s true).
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• Independent reading and sense-making are important activities.
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• Independent reading and sense-making are important activities.

• If something’s important, a lecturer should indicate that by giving class time to it.

• It also gives me an opportunity to be a mind-reader.
Consider this argument about an infinite sum:

Let \[ S = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots. \]
Then \[ xS = x + x^2 + x^3 + x^4 + x^5 + \ldots. \]
So \[ S - xS = 1 \]
i.e. \[ (1 - x)S = 1. \]
So \[ S = \frac{1}{1 - x}. \]

Does this work for every value of \( x \)?
What exactly goes wrong in some cases?
Proof of the sum rule for convergent sequences

Let \((a_n) \to a\) and \((b_n) \to b\).

Let \(\varepsilon > 0\) be arbitrary.

Then \(\exists N_1 \in \mathbb{N} \text{ s.t. } \forall n > N_1, \ |a_n - a| < \varepsilon/2\)

and \(\exists N_2 \in \mathbb{N} \text{ s.t. } \forall n > N_2, \ |b_n - b| < \varepsilon/2\).

Let \(N = \max\{N_1, N_2\}\).

Then \(\forall n > N,\)

\[
| (a_n + b_n) - (a + b) | = | a_n - a + b_n - b |
\leq |a_n - a| + |b_n - b| \text{ by the triangle inequality}
< \varepsilon/2 + \varepsilon/2
= \varepsilon.
\]

So \(\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, \ |(a_n + b_n) - (a + b)| < \varepsilon\).

So \((a_n + b_n) \to a + b\) as required.
Activities: true/false
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• 10 questions: state true or false and, if false, give a counterexample or a brief reason.
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- Shouting out answers (with repeats).
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• Shouting out answers (with repeats).
• Any item in the module can be converted to a true/false question.
A1. The number $\sqrt{47}$ is irrational.

A2. The number $47/225$ has a non-repeating decimal expansion.

A3. The set of even numbers is countably infinite.

A4. For all $x \in \mathbb{N}$, $4|x^3 \Rightarrow 4|x$.

A5. If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ then $x+y \notin \mathbb{Q}$.

A6. If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ then $xy \notin \mathbb{Q}$.

A7. $0.279999\ldots = 0.28$. 
Activities: tests
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- Week 3, 6, 9.
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• Individual and generic feedback provided.
Student feedback
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• Gives us chances during lectures to discuss the module, allowing us to reinforce our understanding of the module.
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• I personally find that I’m rushed for time on the 10% class tests - I always have to guess the last few true/false.
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- The lecturer makes us revise/learn new topics frequently with multiple tests making the later content easier to understand.
- I personally find that I’m rushed for time on the 10% class tests - I always have to guess the last few true/false.
- Sometimes too much time spent trying to get us to interact with each other even if the large majority of the class has nothing to say.
Atmosphere
Atmosphere
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- This is about the mood in the room.
Atmosphere

• This is about the mood in the room.
• The lecturer is the biggest influence on the mood.
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- It’s much nicer to be in a room with 200 challenged but happy people than 200 miserable ones.
- Practices: explicitly value peer support, compliment good attitude and behaviour, notice the mood, draw attention to learning, share the enthusiasm, capitalise on spontaneous humour.
Atmosphere
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- Explicitly value peer support.
Atmosphere

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Atmosphere

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Atmosphere
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\[ A9. \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots \quad \forall x \in \mathbb{R}. \]

\[ A10. \quad \text{While revising I will study for ten hours per day.} \]
Atmosphere

• Capitalise on spontaneous humour.

A9. \( \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots \forall x \in \mathbb{R} \).

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Student feedback
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• Very different to A level maths. Challenging module but certainly satisfying being able to learn all the new concepts, theorems and proofs…
• It was something I had never done before and it made me think of maths in a completely different and new way.
• I like how it makes you want to question everything you know and make you think differently.
Thank you.