Partial Reconstruction Algorithm of a Branching Diffusion with Immigration

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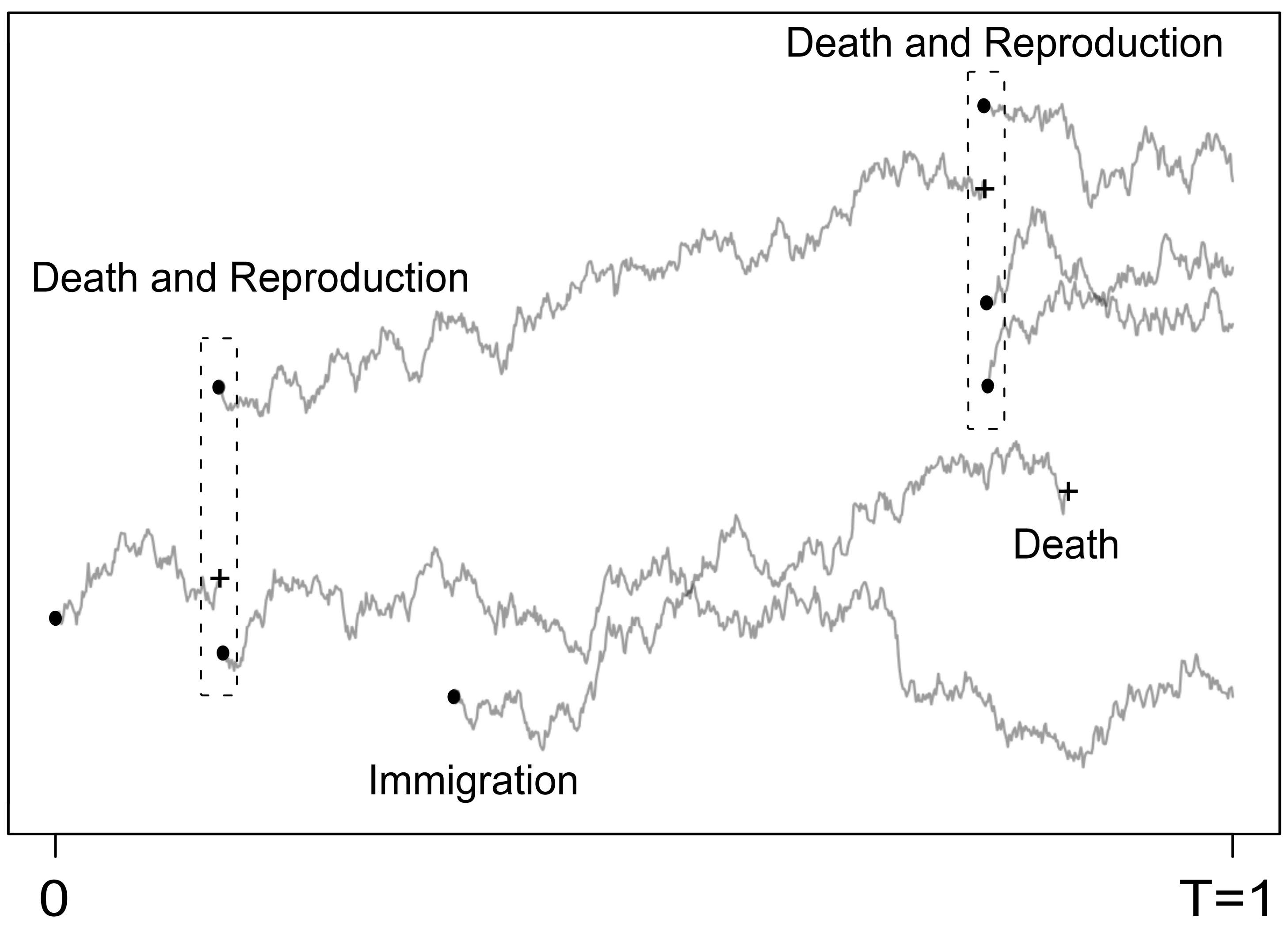
Branching Diffusion with Immigration (BDI)

Systems of finitely many particles:

- One-dimensional particle motion, particles travel independently of each other according to a solution of a diffusion $dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$.
- Branching according to a position-dependent rate: newborn particles are distributed randomly in space.
- Immigration according to a constant rate.

The resulting process $\eta = (\eta_t)_{t\geq 0}$ (values in the configuration space $\mathcal{S} := \bigcup_{l\in\mathbb{N}_0}\mathbb{R}^l$) is called a *branching diffusion with immigration* (BDI).

How does a typical BDI path look like?



Assumptions on the BDI

- Harris recurrence (void configuration as a recurrent atom), the corresponding finite invariant measure is $m(\cdot)$.
- Both the finite invariant measure $m(\cdot)$ and the finite occupation measure

$$\overline{m}(B) := \int_{\mathcal{S}} \mathbf{x}(B)\, m(d\mathbf{x}),\ B \in \mathcal{B}(\mathbb{R}),$$

admit for continuous Lebesgue densities.

Statistics on a BDI

- **What is our statistical aim?**

  Estimation of the diffusion coefficient $\sigma(\cdot)$ by observing the path of the BDI at discrete points of time. We wish to fill a "classical" regression scheme.
- **What is needed?**

  Reconstruction of the underlying BDI path from discrete observations, i.e., at discrete time points with step size Δ.
- **Why is it needed?**

  Observing the path of the BDI at discrete points of time
  - one cannot see the pedigree of the particles.
  - one does not know which particle belongs to which diffusion path.
- **What is special?**

  When reconstructing the underlying BDI path, we are not interested in keeping all information: we only consider "good" observations and reject "bad" ones.

Partial Reconstruction Algorithm

Let $0 < \lambda < 1/2$, let T be a fixed point of time and let $\beta_{i\Delta} = (\beta^1_{i\Delta}, ..., \beta^{l(\beta_{i\Delta})}_{i\Delta})$ be an arrangement of $\eta_{i\Delta} = (\eta^1_{i\Delta}, ..., \eta^{l(\eta_{i\Delta})}_{i\Delta})$ with $0 \leq i \leq \lfloor T/\Delta \rfloor - 1$ and $1 \leq k \leq l(\eta_{i\Delta})$.

- Let $\beta^{[i\Delta,\beta^k_{i\Delta}]}_{(i+1)\Delta}$ be the set of components of $\beta_{(i+1)\Delta}$ whose distance to every $\beta^k_{i\Delta}$ (with $1 \leq k \leq l(\beta_{i\Delta})$) is less than Δ^λ.
- Let D_ε be the set of configurations of $\mathcal{S}$ whose components are at least ε away from each other. Its complement is denoted by $S_\varepsilon := \mathcal{S}\backslash D_\varepsilon$.

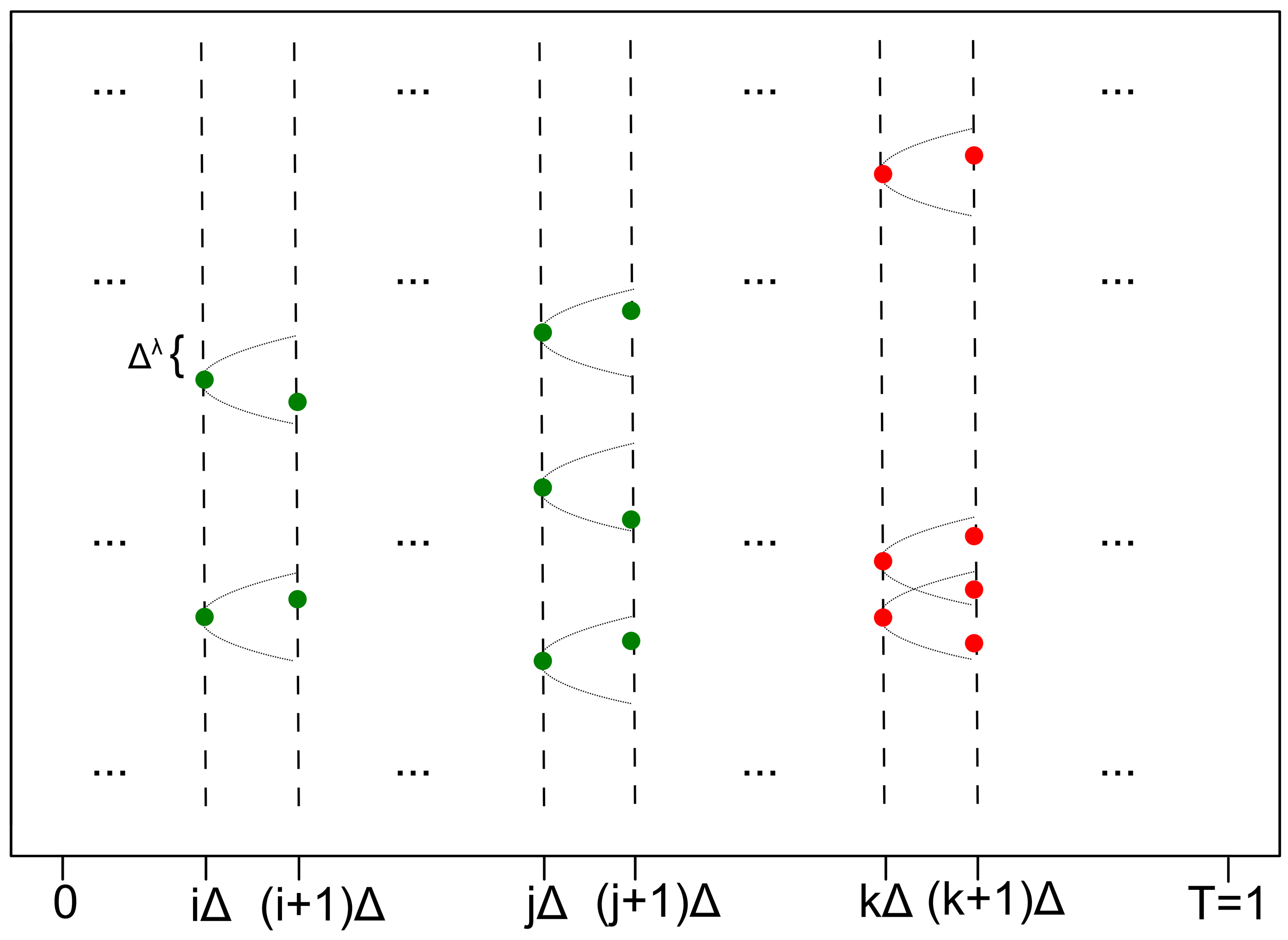
Definition

A pair of successive $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$, $0 \leq i \leq \lfloor T/\Delta \rfloor - 1$, *is called interpretable if there exists an arrangement* $(\beta_{i\Delta}, \beta_{(i+1)\Delta})$ *with*

$$\begin{cases} \beta_{i\Delta} \in D_{4\Delta^\lambda} \quad \text{and} \quad \beta_{(i+1)\Delta} \in D_{2\Delta^\lambda}, \\ \left|\beta^{[i\Delta,\beta^k_{i\Delta}]}_{(i+1)\Delta}\right| = 1 \quad \text{for every} \quad 1 \leq k \leq l(\beta_{i\Delta}). \end{cases}$$

The following plot illustrates this definition:

- Green: Interpretable pairs of successive observations.
- Red: Non-interpretable pairs of successive observations.



Our interpretation yields an assignment which *may be wrong*.

Definition

Let G *be the set of all interpretable pairs* $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$, $0 \leq i \leq \lfloor T/\Delta \rfloor - 1$, *such that the assignment is correct and no component of* $\eta_{i\Delta}$ *branches during time* $(i\Delta, (i+1)\Delta]$. *Call pairs in* G *"properly interpretable".*

Results of the algorithm

Theorem 1

$$1 \geq \mathbf{E}_m\left(\frac{1}{\lfloor T/\Delta\rfloor}\sum_{i=0}^{\lfloor T/\Delta\rfloor-1} \mathbf{1}_G\big((\eta_{i\Delta}, \eta_{(i+1)\Delta})\big)\right) \geq 1 - \mathcal{O}(\Delta^\lambda), \quad as \quad \Delta \to 0.$$

Using nice properties of densities of $m(\cdot)$ and $\overline{m}(\cdot)$ (c.f. Hammer 2012):

Theorem 2

$$m(S_\varepsilon) = \mathcal{O}(\varepsilon), \quad as \quad \varepsilon \to 0.$$

References

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