Volatility estimation from high-frequency observations with irregular errors – Concepts and consequences

or *Improved volatility estimation based on limit order books*

Markus Bibinger, joint work with Moritz Jirak and Markus Reiß

Outline

1. Model & Motivation
2. Volatility estimation based on local order statistics
3. Law of local minima and upper bound
4. Lower bound
5. Application

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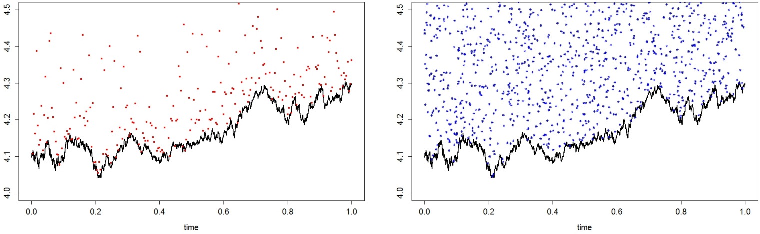
Statistical model



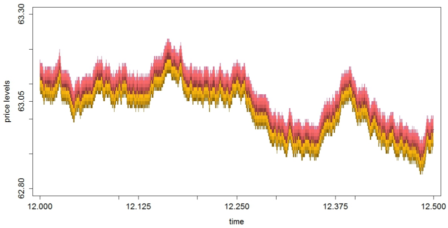
Continuous Itô semi-martingale $X_t = X_0 + \int_0^t a_s\,ds + \int_0^t \sigma_s\,dW_s$, $t \in [0,1]$, $(T_j, \mathcal{Y}_j)$ obs. of Poisson point process on $[0,1]\times\mathbb{R}$ with intensity measure

$$\Lambda(A) = \int_0^1 \int_{\mathbb{R}} \mathbb{1}_A(t,y)\lambda_{t,y}\,dt\,dy, \quad \lambda_{t,y} = n\lambda\mathbb{1}(y \geq X_t).$$

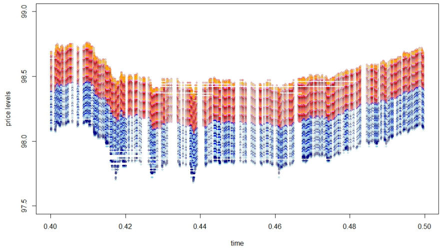
Connatural discrete-time model:

$$Y_i = X_{t_i^n} + \varepsilon_i,\ i = 0,\ldots,n,\ \varepsilon_i \geq 0,\ \varepsilon_i \overset{iid}{\sim} F_\lambda(x) = \lambda x(1+\mathcal{O}(1)).$$

Intra-day order book price dynamics



Order price levels for Facebook asset, 12:00 - 12:30, June 2nd 2014, levels 1-5, bid-ask spread colored in dark red.



Order levels 1-30 and arrivals for AAPL, 12:00 - 12:45, July 28th 2014.

Data by LOBSTER.

Groundwork & Contribution

Principal objective: Recovery of quadratic variation of stochastic boundary X_t.

Main application: Estimation of *integrated volatility* for portfolio and risk management.

Groundwork on volatility estimation:

- For discrete observations $X_{i/n}, i = 0, \ldots, n$, the *realized volatility* satisfies
$$n^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left(X_{\frac{i}{n}} - X_{\frac{(i-1)}{n}}\right)^2 - \int_0^1 \sigma_s^2\,ds\right) \rightsquigarrow \mathrm{N}\left(0, 2\int_0^1 \sigma_s^4\,ds\right),$$
and is *asymptotically efficient*.

- However, the direct observation model does not accurately fit high-frequency data.

- Prominent *microstructure noise model*: $Y_i = X_{t_i^n} + \varepsilon_i$ with ε_i i.i.d., $\mathbb{E}[\varepsilon_i] = 0$.

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Groundwork on volatility estimation:

- For discrete observations $X_{i/n}, i = 0, \ldots, n$, the *realized volatility* is *asymptotically efficient*.
- However, the direct observation model does not accurately fit high-frequency data.
- Prominent *microstructure noise model*: $Y_i = X_{t_i^n} + \varepsilon_i$ with ε_i i.i.d., $\mathbb{E}[\varepsilon_i] = 0$. Efficient estimator by Bibinger et al (2014) for $\varepsilon_i \stackrel{iid}{\sim} N(0, \eta^2)$ satisfies
  $n^{\frac{1}{4}} \left(\hat{IV} - \int_0^1 \sigma_s^2 \, ds \right) \rightsquigarrow N\left(0, 8\eta \int_0^1 \sigma_s^3 \, ds\right)$.

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Groundwork on volatility estimation:

- For discrete observations $X_{i/n}, i = 0, \ldots, n$, the *realized volatility* is *asymptotically efficient*.
- However, the direct observation model does not accurately fit high-frequency data.
- Prominent *microstructure noise model*: $Y_i = X_{t_i^n} + \varepsilon_i$ with ε_i i.i.d., $\mathbb{E}[\varepsilon_i] = 0$. Efficient estimator by Bibinger et al (2014).
- Apply estimators to *which* time series of prices (micro prices, traded prices, etc.)?

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Construction of an estimator

Partition the unit interval into $h_n^{-1} \in \mathbb{N}$ equi-spaced bins $\mathcal{T}_k^n = [kh_n, (k+1)h_n), k = 0, \ldots, h_n^{-1} - 1$, $nh_n \in \mathbb{N}$, $h_n \to 0$.
Parametric estimation theory motivates bin-wise minima

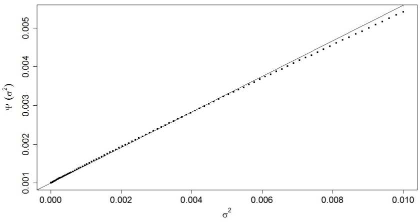
$$m_{n,k} = \min_{i \in \mathcal{I}_k^n} Y_i \ , \ \ \mathcal{I}_k^n = \{kh_n n, kh_n n + 1, \ldots, (k+1)h_n n - 1\},$$

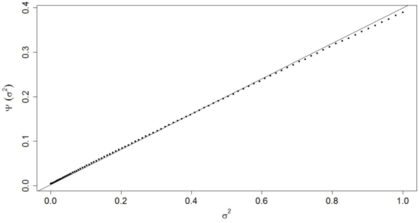
$$m_{n,k} = \min_{T_j \in \mathcal{T}_k^n} \mathcal{Y}_j \, .$$

as estimators of X_{kh_n}. $\mathbb{V}\mathrm{ar}(\min_{i \in \mathcal{I}_k^n} \varepsilon_i) \propto (n\lambda h_n)^{-2}$, so locally constant signal approximation $X_t = X_{kh_n} + \mathcal{O}_{\mathbb{P}}(h_n^{1/2})$ on $\mathcal{T}_k^n$ is only admissible when $h_n^{1/2} = o((n\lambda h_n)^{-1})$.
Optimal rate attained when

$$h_n = \mathcal{K}^{\frac{2}{3}} (n\lambda)^{-\frac{2}{3}}, \mathcal{K} > 0, \ nh_n \propto n^{\frac{1}{3}} \lambda^{-\frac{2}{3}} \, .$$

The function Ψ





Introduce for $X_t = X_{kh_n} + \sigma_{kh_n} \int_{kh_n}^{t} dW_s$ on $\mathcal{T}_k^n$ in PPP-model:

$$\Psi\left(\sigma^2_{kh_n}\right) = h_n^{-1}\mathbb{E}\Big[\big(m_{n,k} - m_{n,k-1}\big)^2\Big], k = 1, \ldots, h_n^{-1} - 1,$$

an invertible function, MC approximation above for $\mathcal{K} = 32$.

$$\Psi^{-1}\left(\sum_{k=(l-1)r_n^{-1}/2+1}^{lr_n^{-1}/2} \big(m_{n,2k} - m_{n,2k-1}\big)^2 2h_n^{-1} r_n\right) \approx \sigma^2_{lr_n^{-1}h_n},$$

where $r_n^{-1}h_n$ is a coarse grid size with $r_n h_n^{-1}, r_n^{-1} \in 2\mathbb{N}$.

The estimator based on local minima

$$\widetilde{IV}_n^{h_n,r_n} = \sum_{l=1}^{r_n h_n^{-1}} \Psi^{-1}\Bigg(\sum_{k=(l-1)r_n^{-1}/2+1}^{l r_n^{-1}/2} \big(m_{n,2k} - m_{n,2k-1}\big)^2 \, 2h_n^{-1} r_n \Bigg) h_n r_n^{-1} .$$

In the regression-type model

$$\Psi_n\big(\sigma^2_{kh_n}\big) = h_n^{-1}\mathbb{E}\Big[\big(m_{n,k} - m_{n,k-1}\big)^2\Big], k = 1, \ldots, h_n^{-1} - 1 ,$$

with a sequence $\Psi_n \to \Psi$. Estimator

$$\widehat{IV}_n^{h_n,r_n} = \sum_{l=1}^{r_n h_n^{-1}} \Psi_n^{-1}\Bigg(\sum_{k=(l-1)r_n^{-1}/2+1}^{l r_n^{-1}/2} \big(m_{n,2k} - m_{n,2k-1}\big)^2 \, 2h_n^{-1} r_n \Bigg) h_n r_n^{-1} .$$

For $\sigma_t = \sigma =$ const., use $\widehat{IV}_n^{h_n,h_n}$. For Lipschitz σ_t balance approximation error $r_n^{-1} h_n$ with second order term on each coarse interval of order $r_n \Rightarrow r_n \propto h_n^{1/2} = (n\lambda)^{-1/3}$.

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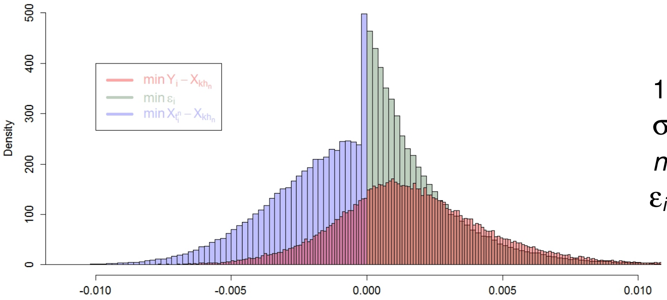
The law of bin-wise minima



10^5 bins, $\sigma = 1$, nh_n=100, $\varepsilon_i \sim \text{Exp}(5)$

Rewrite $m_{n,k} - m_{n,k-1} = \mathcal{R}_{n,k} - \mathcal{L}_{n,k}$, where $\mathcal{R}_{n,k} = m_{n,k} - X_{kh_n}$, $\mathcal{L}_{n,k} = m_{n,k-1} - X_{kh_n}$. For $X_t = X_{kh_n} + \sigma \int_{kh_n}^t dW_s$, invoke time-reversibility to see that $\mathcal{R}_{n,k}, \mathcal{L}_{n,k}$, $k = (l-1)r_n^{-1}+1, \ldots, lr_n^{-1}$, are all identically distributed and independent, except $\mathcal{R}_{n,k}$ and $\mathcal{L}_{n,k+1}$. Infer $\Psi(\sigma_{kh_n}^2)h_n = 2\,\mathbb{V}\text{ar}(\mathcal{R}_{n,k})$, and similarly for Ψ_n.

Connection to Brownian excursion areas

Proposition *Choose* $h_n = \mathcal{K}^{2/3}(n\lambda)^{-2/3}$. *Consider* $t \in \mathcal{T}_k^n$ *and* $X_t = X_{kh_n} + \int_{kh_n}^t \sigma\, dW_s, t \in \mathcal{T}_k^n$. *Then in the PPP-model for all* $x \in \mathbb{R}$:

$$\mathbb{P}\Big(h_n^{-1/2}\mathcal{R}_{n,k} > x\sigma\Big) = \mathbb{E}\Big[\exp\Big(-\mathcal{K}\sigma\int_0^1 (x+W_t)_+ dt\Big)\Big].$$

In the regression-type model for all $x \in \mathbb{R}$:

$$\lim_{n\to\infty}\mathbb{P}\Big(h_n^{-1/2}\mathcal{R}_{n,k} > x\sigma\Big) = \mathbb{E}\Big[\exp\Big(-\mathcal{K}\sigma\int_0^1 (x+W_t)_+\, dt\Big)\Big].$$

Laplace transforms of such expressions can be obtained via *Feynman–Kac formula*. It draws a connection between measures on the path space to parabolic PDEs as the transitional probability p for Brownian motion obeys

$$\partial_t p(t,x;s,y) = \tfrac{1}{2}\partial_x^2 p(t,x;s,y)\,,$$
$$-\partial_s p(t,x;s,y) = \tfrac{1}{2}\partial_y^2 p(t,x;s,y)\,.$$

Result with Feynman–Kac formula

The Laplace transform (in t) of

$$\mathbb{E}\Big[\exp\Big(-\sqrt{2}\theta\int_0^t (W_s+x)_+\,ds\Big)\Big],\ \theta\in\mathbb{R},$$

is derived as solution of

$$\frac{d^2\zeta}{dx^2}=2s\zeta-2\theta^{2/3},x<0,\ \frac{d^2\zeta}{dx^2}=2(\sqrt{2}\theta x+s)\zeta-2\theta^{2/3},x>0.$$

With the *Airy function* Ai and *Scorer function* Gi

$$\mathrm{Ai}(x)=\pi^{-1}\int_0^\infty \cos\left(t^3/3+xt\right)dt,\ \mathrm{Gi}(x)=\pi^{-1}\int_0^\infty \sin\left(t^3/3+xt\right)dt\ :$$

$$\mathbb{E}\left[\int_0^\infty \exp\left(-st-\sqrt{2}\theta\int_0^t (W_s+x)_+\,ds\right)dt\right]=\theta^{-\frac{2}{3}}\zeta_s(x,\theta),$$

$$\zeta_{s,+}(x,\theta)=A_s\,\mathrm{Ai}\left(\sqrt{2}\theta^{\frac{1}{3}}x+\theta^{-\frac{2}{3}}s\right)+\pi\mathrm{Gi}\left(\sqrt{2}\theta^{\frac{1}{3}}x+\theta^{-\frac{2}{3}}s\right)$$

$$\zeta_{s,-}(x,\theta)=B_s\exp\left(\sqrt{2s}x\right)+s^{-1}\theta^{2/3}.$$

Convergence rate of the estimator

Assumption *The drift a_s is bounded and Borel-measurable, the volatility σ_t is a Lipschitz function, $\sigma_t > 0$. The constant $\mathcal{K}$ of h_n is chosen sufficiently large.*

Theorem *For $h_n = \mathcal{K}^{2/3}(n\lambda)^{-2/3}$ and $r_n = \kappa n^{-1/3}$, $\kappa > 0$, the estimator based on the PPP-model satisfies*

$$\left(\widetilde{IV}_n^{h_n,r_n} - \int_0^1 \sigma_s^2\,ds\right) = \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{1}{3}}\right).$$

Corollary *The estimator based on the regression-type model satisfies*

$$\left(\widehat{IV}_n^{h_n,r_n} - \int_0^1 \sigma_s^2\,ds\right) = \mathcal{O}_{\mathbb{P}}\left(n^{-\frac{1}{3}}\right).$$

Outline

The lower bound for the minimax rate

We show for the PPP-model that in the parametric experiment, $X_t = \sigma\, dW_t$, $t \in [0,1]$, $\sigma > 0$ unknown, the optimal rate of convergence is $n^{-1/3}$ in minimax sense. This serves a fortiori as a lower bound for the general nonparametric case.

Theorem

For any sequence of estimators $\hat{\sigma}_n^2$ of $\sigma^2 \in (0,\infty)$ from the parametric PPP-model for each $\sigma_0^2 > 0$, the local minimax lower bound is

$$\exists \delta > 0: \liminf_{n\to\infty} \inf_{\hat{\sigma}_n} \max_{\sigma^2 \in \{\sigma_0^2, \sigma_0^2 + \delta n^{-1/3}\}} \mathbb{P}_{\sigma^2}\left(|\hat{\sigma}_n^2 - \sigma^2| \geq \delta n^{-1/3}\right) > 0,$$

where the infimum extends over all estimators $\hat{\sigma}_n$ based on the PPP-model with $\lambda = 1$ and $X_t = \sigma W_t$. The law of the latter is denoted by $\mathbb{P}_{\sigma^2}$.

Sketch of information-theoretic proof

First reduction: Decompose in sum of two independent PPPs:

$$\mathrm{PPP}_{\Lambda} = \mathrm{PPP}_{\Lambda_r} + \mathrm{PPP}_{\Lambda_s} \text{ with } \Lambda = \Lambda_r + \Lambda_s\,,$$
$$\lambda_r(t,y) = n\Big(\big(b^{-1}(y - X_t)_+\big)^2 \wedge 1\Big), b > 0, \lambda_s = \lambda - \lambda_r\,.$$

Provide more information by $(T_j^s, X_{T_j^s})_{j\geq 1}$ instead of $(T_j^s, Y_j^s)_{j\geq 1}$.
Second reduction: Conditional on (T_j^s) observations $(T_i^r, \mathcal{Y}_i^r)$, $T_i^r \in [T_{j-1}^s, T_j^s)$ form for each j independent PPPs on $[0, T_j^s - T_{j-1}^s]$ with intensities

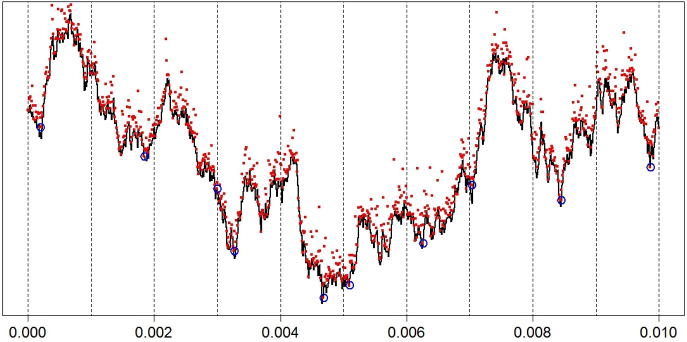
$$\lambda^j(t,y) = n\Big(b^{-1}\Big(y - \sigma B_t^{0, T_j^s - T_{j-1}^s}\Big)_+ \wedge 1\Big)\,,$$

with a Brownian bridge denoted by $B^{0,T}$.
For this more informative experiment standard bounds for the Hellinger distance imply the Theorem, $b \propto n^{-1/3}$.

Outline

Example for one realised path

Regression model, $\sigma = 1$, exponential errors with $\lambda^{-1} = 0.005$, $n = 100000$, $h_n = 0.001$, $nh_n = 100$.



First order approximation of Ψ

When $h_n = \mathcal{K}^{2/3}\lambda^{-2/3}n^{-2/3}$ and $\lambda^{-1} \ll \sigma_t$ for all t, or $\mathcal{K}$ large, the local minima are predominantly determined by $\min_{i\in\mathcal{I}_k^n} X_{t_i^n}$.
High signal-to-noise ratios found for high-frequency data in empirical studies.
Joint law of end-point of $Z = \int \sigma dW$ and its minimum:

$$\mathbb{P}\Big(\min_{0\le s\le t} Z_s < m, Z_t \ge w\Big) = \int_{-\infty}^{2m-w} (2\pi\sigma^2)^{-1/2}\exp\{-1/(2\sigma^2)z^2\}dz\,,$$

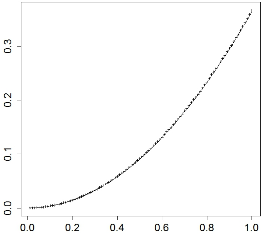
$$g(m,w) = \tfrac{2(w-2m)}{\sigma^3\sqrt{2\pi}}\exp\{-1/(2\sigma^2)(2m-w)^2\}, m\in(-\infty,0], w\in[m,\infty)\,.$$

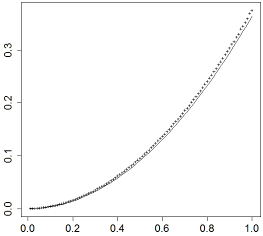
This yields for $k = \lfloor th_n^{-1}\rfloor$, $m_{n,k} = \min_{i\in\mathcal{I}_k^n} X_{t_i^n}$:

$$\lim_{n\to\infty} h_n^{-1/2}\mathbb{E}[\mathcal{L}_{n,k}] = -\sqrt{(2/\pi)}\sigma_t\,, \lim_{n\to\infty} h_n^{-1}\mathbb{E}[\mathcal{L}_{n,k}^2] = \sigma_t^2\,,$$

$$\lim_{n\to\infty} h_n^{-1}\mathbb{E}[\mathcal{L}_{n,k+1}\mathcal{R}_{n,k}] = \tfrac{1}{2}\sigma_t^2, \Rightarrow \Psi(\sigma_t^2) = 2\sigma_t^2\frac{(\pi-2)}{\pi}\,.$$

Accuracy of first-order approximation





Comparison of $\sum_{k=1}^{h_n^{-1}/2}(m_{n,2k}-m_{n,2k-1})^2$ and $\sigma^2(\pi-2)/\pi$ for $\sigma\in[0,1]$, $n=100000, nh_n=100$; $\lambda^{-1}=0.005$ (top) and $\lambda^{-1}=0.05$ (bottom).

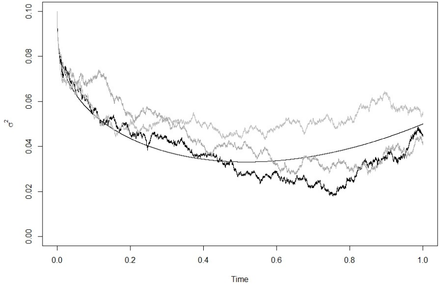
Conclude simple estimator

$$\widehat{IV}_{n,app}^{h_n}=\frac{\pi}{\pi-2}\sum_{k=1}^{h_n^{-1}/2}(m_{n,2k}-m_{n,2k-1})^2.$$

Monte Carlo simulations: Setup

Simulate $Y_i = X_{i/n} + \varepsilon_i, i = 0, \ldots, n,$.

1. $\sigma_t^2 = 0.1\left(1 - 0.4\sin\left(\frac{3}{4}\pi t\right)\right),\; t \in [0,1]$, drift $a = 0.1$.
2. $\sigma_t^2 = \left(\int_0^t c \cdot \rho\, dW_s + \int_0^t \sqrt{1-\rho^2} \cdot c\, dW_s^{\perp}\right) \cdot \tilde{\sigma}_t,$
   $W^{\perp}$ Brownian motion independent of W, $c = 0.05$, $\rho = 0.5$, $a = 0.1$ and seasonality function $\tilde{\sigma}_t = 0.1\left(1 - t^{\frac{1}{3}} + 0.5 \cdot t^2\right)$.



Monte Carlo simulations: Results

$n = 100000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
500	200	0.0015	0.1031	0.00	0.0332	0.1171	0.93
1000	100	0.0068	0.0542	0.08	0.0407	0.0518	3.10
10000	10	0.1393	0.0060	76.40	0.8688	0.0085	98.88
$n = 10000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	100	0.0020	0.1227	0.00	0.0179	0.1200	0.26
500	20	0.0183	0.0219	1.35	0.0555	0.0239	11.43
1000	10	0.0392	0.0116	11.75	0.1401	0.0138	58.72

Results for $\hat{IV}_{n,app}^{h_n}$ in setup 1. Bias rescaled with $n^{1/3}$, variance with factor $n^{2/3}$. Based on *first order approximation*.
Results for setup 2 equally well, $\hat{IV}_{n,app}^{h_n}$ comes close but does not attain minimal MSE because of the bias.

Monte Carlo simulations: Results

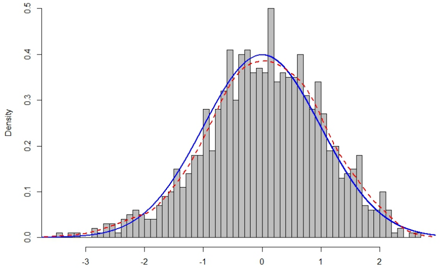
$n = 100000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	r_n^{-1}	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
500	100	-0.0069	0.1051	0.04	-0.0194	0.1049	0.36
1000	100	0.0095	0.0528	0.17	-0.0148	0.0571	0.38
10000	100	0.0064	0.0056	0.72	0.0066	0.0077	0.56
$n = 10000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	r_n^{-1}	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	20	-0.0078	0.1168	0.05	0.0029	0.1199	0.00
500	100	-0.0136	0.0237	0.77	-0.0008	0.0257	0.00
1000	100	0.0086	0.0125	0.59	0.0109	0.0135	0.87

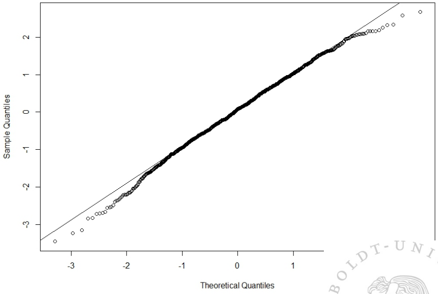
Results for $\hat{IV}_n^{h_n,r_n}$ in setup 1. Bias rescaled with $n^{1/3}$, variance with factor $n^{2/3}$. Results for setup 2 equally well, also different error distributions considered in report. Above choice of r_n slightly better than $r_n = \sqrt{h_n}$.

Monte Carlo approximation of Ψ_n employed.

Monte Carlo empirical distribution

Parametric setup $\sigma = 1$. Empirical distribution of $(\widehat{\mathbb{V}\mathrm{ar}}(\hat{\sigma}_n^2))^{-1/2}\,\hat{\sigma}_n^2$ for 1000 iterations, , $\lambda^{-1} = 0.005$, $n = 100000, h_n = 1000$.





Summary & Outlook

Summary:

- *Improved* volatility estimation based on order prices.
- We obtain $n^{1/3}$ as *optimal convergence rate*.
- First applications promising.
- First data applications using orders and traded prices support the idea of the *same latent efficient price* and its volatility recovered with methods based on microstructure noise model with centred and one-sided errors, respectively.

Summary & Outlook

Outlook:

- Strive for stable CLT in model where the volatility is a semi-martingale.
- Design an explicit feasible estimator.
- Application: Validate model using all available information from bids, asks, trades.

Literature

Bibinger, M., Jirak, M., Reiß, M., (2014).
Improved volatility estimation based on limit order books.
arXiv:1408.3768

Bibinger, M., Jirak, M., Reiß, M., (2014).
Applying volatility estimators based on limit order books.
technical report, available under `www.mathematik.hu-berlin.de/for1735/Publ/application.pdf`.

Thank you for your attention!