# Nonparametric estimation in a mixed-effect Ornstein-Uhlenbeck model

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# Content

# Mixed-effect Ornstein-Uhlenbeck model

Observations: $(X_j(t), 0 \leq t \leq T)$ $j = 1, \ldots, N$, $N$ processes described by

$$\begin{cases} dX_j(t) & = \left(\phi_j - \frac{X_j(t)}{\alpha}\right) dt + \sigma dW_j(t) \\ X_j(0) & = x_j \end{cases}$$

$\rightarrow$ it models the variability along time for each subject.

- $(W_j)_{1 \leq j \leq N}$ are $N$ independent standard Wiener processes.
- $(\phi_j)_{1 \leq j \leq N}$ are $N$ unobserved *i.i.d.* r.v. with density $f$: random effect of individual $j$.
- $(\phi_j)_{1 \leq j \leq N}$ and $(W_j)_{1 \leq j \leq N}$ are independent.
- $(x_1, \ldots, x_N)$ are known values.
- $T$ in fixed, known.
- The positive constants $\sigma$ and $\alpha$ are supposed to be known.

→ When $t$ is fixed: due to the independence of the $\phi_j$ and the $W_j$, the $X_j(t)$ are $N$ *i.i.d.* r. v.

$$X_j(t) = X_j(0)e^{-t/\alpha} + \phi_j\alpha(1 - e^{-t/\alpha}) + \sigma e^{-t/\alpha}\int_0^t e^{s/\alpha}dW_j(s).$$

→ Differences between observations are due to the realization of both the $W_j$ and $\phi_j$.

→ However, the $N$ trajectories $(X_j(t), 0 \leq t \leq T), j = 1, \ldots, N$ are *i.i.d.*

$X_j$ represents the behaviour of one individual and $\phi_j$ describes the individual specificity.

**Goal:** to estimate in a nonparametric way the density $f$ of the random effects.

- Parametric approach: Gaussian assumption (*c.f.* *e.g* Genon-Catalot and Larédo, (2013), Donnet and Samson, (2008), Delattre *et al.*, (2013)).
- Nonparametric: Comte *et al.*, (2013), for large $T$. Not efficient when $T$ is small.

**Proposal:** a new nonparametric estimator, built by **deconvolution**, depending on **two parameters**, selected in a **data-driven way**.

# Notations

Let us consider $f$ and $g$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$:

- $\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2 dx$.
- The Fourier transform of $f$: $f^*(x) = \int_{\mathbb{R}} e^{iux} f(u) du$ for all $x \in \mathbb{R}$.
- The convolution product of $f$ and $g$: $f \star g(x) = \int_{\mathbb{R}} f(x-y) g(y) dy$.

We assume

$$\textbf{(A)}\ f \in L^2(\mathbb{R}),\ f^* \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

# Construction of the estimator: deconvolution steps

$$dX_j(t) = \left(\phi_j - \frac{X_j(t)}{\alpha}\right) dt + \sigma dW_j(t), \quad X_j(0) = x_j$$

For $j = 1, \ldots, N$, $\tau \in ]0, T]$, estimators of the $\phi_j$

$$Z_{j,\tau} := \frac{X_j(\tau) - X_j(0) - \int_0^\tau (-\frac{X_j(s)}{\alpha} ds)}{\tau}.$$

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Notice that

$$Z_{j,\tau} = \phi_j + \frac{\sigma}{\tau} W_j(\tau).$$

When $\tau$ is fixed: the two members of the sum are independent, thus $(Z_{j,\tau})_{j=1,\ldots,N}$ are *i.i.d.*, and

$$f_{Z_\tau}(u) = f \star f_{\frac{\sigma}{\tau} W_j(\tau)}(u).$$

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Fourier transform under **(A)**

$$f^*_{Z_\tau}(u) = f^*(u) f^*_{\frac{\sigma}{\tau} W_j(\tau)}(u) \;\Leftrightarrow\; f^*(u) = f^*_{Z_\tau}(u) e^{u^2 \sigma^2 / 2\tau}.$$

## Cut-off choice

Fourier inversion

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} f^*_{Z_\tau}(u) e^{\frac{u^2\sigma^2}{2\tau}} du.$$

Estimator of $f^*_{Z_\tau}(u)$: $\widehat{f}^*_{Z_\tau}(u) = (1/N) \sum_{j=1}^{N} e^{iuZ_{j,\tau}}$.

But: integrability of $\widehat{f}^*_{Z_\tau}(u) e^{u^2\sigma^2/2\tau}$ no more ensured → cut-off.

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$$\widehat{f}_\tau(x) = \frac{1}{2\pi}\int_{-\sqrt{\tau}}^{\sqrt{\tau}} e^{-iux}\frac{1}{N}\sum_{j=1}^N e^{iuZ_{j,\tau}} e^{\frac{u^2\sigma^2}{2\tau}} du.$$

$\rightarrow$ Problem when $\tau$ is small.

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→ Problem when $\tau$ is small. We introduce a new cut-off parameter $s$:

$$\widehat{f}_{s,\tau}(x) = \frac{1}{2\pi}\int_{-s\sqrt{\tau}}^{s\sqrt{\tau}} e^{-iux}\frac{1}{N}\sum_{j=1}^N e^{iuZ_{j,\tau}} e^{\frac{u^2\sigma^2}{2\tau}} du.$$

To simplify the theoretical study, we replace $s\sqrt{\tau}$ by a new parameter $m$.

Resulting estimator: $\widetilde{f}_{m,s}$, when $m^2/s^2 \in ]0, T]$,

$$\widetilde{f}_{m,s}(x) = \frac{1}{2\pi} \int_{-m}^{m} e^{-iux} \frac{1}{N} \sum_{j=1}^{N} e^{iuZ_{j,m^2/s^2}} e^{\frac{u^2 \sigma^2 s^2}{2m^2}} du$$

with $m$ and $s$ in two finite sets $\mathcal{M}$ and $\mathcal{S}$.

## Study of the mean integrated squared error (MISE):

Decomposition $\mathbb{E}\left[\|\widetilde{f}_{m,s} - f\|^2\right] = \|f - \mathbb{E}[\widetilde{f}_{m,s}]\|^2 + \mathbb{E}\left[\|\widetilde{f}_{m,s} - \mathbb{E}[\widetilde{f}_{m,s}]\|^2\right]$.

Definition $f_m$ is defined by $f_m^* := f^* \mathbf{1}_{[-m,m]}$.

**Proposition**

*Under **(A)**, $\mathbb{E}[\widetilde{f}_{m,s}] = f_m$ and we have*

$$\mathbb{E}\left[\|\widetilde{f}_{m,s} - f\|^2\right] \leq \|f_m - f\|^2 + \frac{m}{\pi N} \int_0^1 e^{\sigma^2 s^2 v^2} dv.$$

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**Bias term**: decreases when $m$ increases, independent of $s$

$$\|f_m - f\|^2 = \frac{1}{2\pi} \int_{|u| \geq m} |f^*(u)|^2 du.$$

**Variance term**: increases with $m$ and $s$

$$\frac{m}{\pi N} \int_0^1 e^{\sigma^2 s^2 v^2} dv.$$

$\rightarrow$ Bounded as soon as $s$ is bounded.

# Finite collections

→ We want to choose the best couple $(m, s)$: the one realizing the bias-variance compromise.

$$\mathcal{S} := \{s_l = \frac{1}{2^l}\frac{2}{\sigma},\ 1/2^{P-1} \leq \sigma s_l \leq 2,\ l = 0, \ldots, P\}$$

$$\mathcal{M} := \{m = \frac{\sqrt{k\Delta}}{\sigma},\ k \in \mathbb{N}^*,\ 0 < m \leq N\}$$

with $0 < \Delta < 1$ a small step to be fixed.

$$\mathcal{C} := \{(m, s) \in \mathcal{M} \times \mathcal{S},\ m^2/s^2 \leq T\}.$$

## New criterion extended from the Goldenshluger and Lepski's method

Consider $(m, s) \in \mathcal{C}$.

Penalty function

$$\text{pen}(m, s) = \kappa \frac{m}{N} e^{\sigma^2 s^2},$$

where $\kappa$ is a numerical constant to be calibrated.

Criterion

$$\Gamma_{m,s} \quad = \quad \max_{(m',s') \in \mathcal{C}} \left( \| \widetilde{f}_{m',s'} - \widetilde{f}_{(m',s') \wedge (m,s)} \|^2 - \text{pen}(m', s') \right)_+$$

where $(m', s') \wedge (m, s) := (m' \wedge m, s' \wedge s)$.

Selection

$$(\widetilde{m}, \widetilde{s}) = \arg \min_{(m,s) \in \mathcal{C}} \{ \Gamma_{m,s} + \text{pen}(m, s) \}.$$

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**Selection**

$$(\widetilde{m}, \widetilde{s}) = \arg \min_{(m,s) \in \mathcal{C}} \{\Gamma_{m,s} + \text{pen}(m, s)\}.$$

**Lemma**

*There is a constant $C'$ depending on $\|f\|$, $\sigma$, $\Delta$, and $P+1$ the cardinality of $\mathcal{S}$, such that:*

$$\mathbb{E}[\Gamma_{m,s}] \leq 18\|f - f_m\|^2 + \frac{C'(P+1)}{N}.$$

# Main non-asymptotic result on the final estimator: oracle type inequality

**Theorem (D. (2014))**

*Under **(A)**, consider the estimator $\widetilde{f}_{\widetilde{m},\widetilde{s}}$, there exists $\kappa_0$ a numerical constant such that, for all penalty constant $\kappa \geq \kappa_0$,*

$$\mathbb{E}[\|\widetilde{f}_{\widetilde{m},\widetilde{s}} - f\|^2] \leq C \inf_{(m,s)\in\mathcal{C}} \left\{\|f - f_m\|^2 + \text{pen}(m,s)\right\} + \frac{C'(P+1)}{N}$$

*where $C > 0$ is a numerical constant as soon as $\kappa$ is fixed and $C'$ is the previous constant of Lemma.*

Automatic realisation of the bias-penalty compromise.

We choose the two parameters in an adaptive way, thus this gives more flexibility in the choice of the estimator.

## Numerical study

- Exact simulation of the processes.
- Discretization, time step $\delta$, small (500 to 2000 observations).
- Choice of parameters and designs (ex: $T = 0.3,\ 10,\ 100,\ 300$)
- Calibration $\kappa = 0.3$.
- $\Delta = 0.08$.

# Study on simulated data with $N = 240$, $T = 0.3, \delta = 0.00015$, $\sigma = 0.0135$, $\alpha = 0.039$

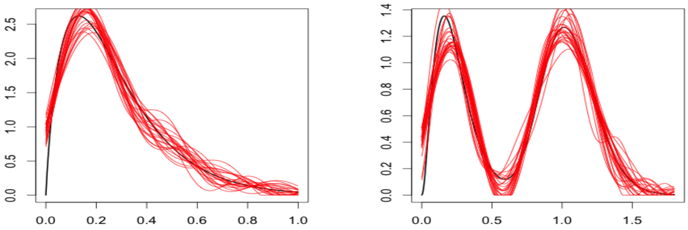


Figure : - 25 estimators $\widetilde{f}_{\widetilde{m},\widetilde{s}}$, - the true density $f$: gamma and mixed gamma

Table : Empirical mean integrated squared error, computed from 100 simulated data sets

| | $f$ gamma | $f$ mixed-gamma |
|---|---|---|
| $\widetilde{f}_{\widetilde{m},\widetilde{s}}$ | 0.068 | 0.038 |
| Oracle | 0.041 | 0.029 |

# Application on a neuronal database

Interspikes interval (ISI) measures: measurements along time of the membrane potential in volts [V] of one single neuron, between the spikes.

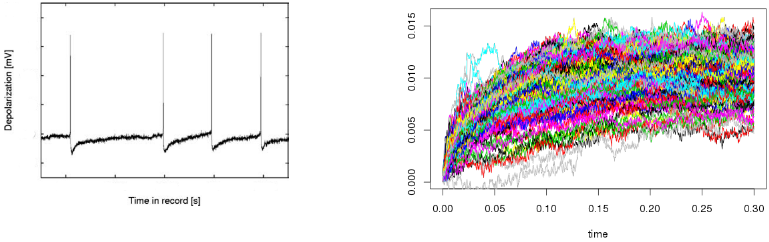


Figure : Left: Membrane potential, right: the 240 observed trajectories

→ **We consider the observations as independent realizations of our model.**
→ **Picchini *et al.* (2010) proves that the Ornstein-Uhlenbeck model with one random effect fits better the data than without.**

## Parameters values

- $T = 0.3$, time step $\delta = 0.00015$ [s].
- The initial voltage = the resting potential: $x_j = 0$.
- The diffusion coefficient is fixed $\sigma = 0.0135$ $[\mathtt{V}/\sqrt{\mathtt{s}}]$ (estimated in Picchini *et al* (2010) )
- $\alpha$ [s]: time constant of the neuron $\alpha = 0.039$ [s] (estimated in Lansky *et al* (2006))

$\rightarrow \phi_j$ is the local input that neuron receives during the $j^{\text{th}}$ ISI.

$\rightarrow$ Estimation of $f$ obtained in Picchini *et al* (2010) under Gaussian assumption: $\mathcal{N}(0.278, 0.041^2)$.

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We although represent the adaptive kernel estimator associate to the r.v. $Z_{j,T}$,

$$\widehat{f}_h(x) = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{h}K\left(\frac{x - Z_{j,T}}{h}\right)$$

where we choose the bandwidth $\widehat{h}$ among a collection, with a data-driven Lepski's procedure developed in the article.

# Estimated density $f$ on a neuronal database

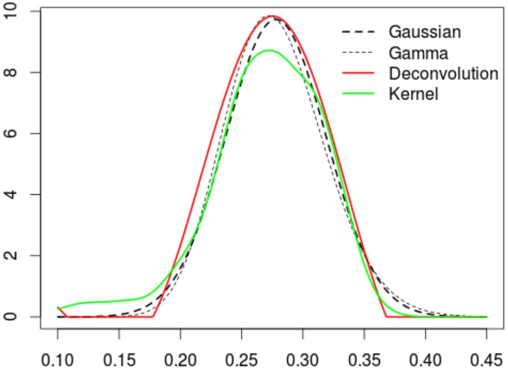


Figure : – $\widehat{f}_{\widehat{h}}$, – $\widetilde{f}_{\widetilde{m},\widetilde{s}}$, - - the density from Picchini *et al* (2010) $\mathcal{N}(0.278, 0.041^2)$ and ... the density $\Gamma(46.3, 0.006)$

## Conclusions

- More precise estimation instead of parametric assumption. Can be used to simulate the $\phi_j$.
- This new parameter $s$ generalizes the results of Comte *et al* (2013) even if $T$ is large.
- Selection procedure of two parameters which can be adapted in other cases.

## Further works

- Remark: the procedure can be written with a drift $b(x) + \phi_j$ with $b$ satisfying assumption but not necessary linear.
- Add a new random effect: $\alpha$.
- Solve the problem when $\sigma(x) \neq \sigma_1$ without assuming $\sigma(x) < \sigma_1$.

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Thank you for your attention.