Quadratic Variation of High Dimensional Itô Processes

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joint work with Mark Podolskij

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Section 1

Setting

p-dimensional Itô process without drift

Stochastic process of the form

$$X_t = X_0 + \int_0^t f_s dW_s$$

where W is a p-dimensional Brownian motion.

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- The RCV is an estimator for the quadratic variation (in 1)

$$[X] = \int_0^1 f_s f_s^* ds.$$

- $[X]_p^n \xrightarrow{\mathbb{P}} [X]$ if $n \to \infty$.

$$[X]_p^n = \sum_{i=1}^{n} \Delta_i^n X \, \Delta_i^n X^*$$

What happens if the dimension of the process p and the number of observations n both are large but of the same order of magnitude? We investigate the behavior of $[X]_p^n$ when $p \to \infty$, $n \to \infty$, and $n/p \to c \in (0, \infty)$.

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- High dimensional random matrix theory provides the framework for this investigation.
- We restrict ourselve to processes with volatility process of the form

$$f_t = \sum_{l=1}^{m} T_l 1_{[t_{l-1}, t_l)}(t) \qquad (1)$$

where $0 = t_0 < \cdots < t_m = 1$, and $T_1, \ldots, T_m$ are $p \times p$ nonrandom matrices.

It is straightforward to show that

$$[X]_p^n \stackrel{d}{=} \frac{1}{n} \sum_{l=1}^{m} T_l Y_l Y_l^* T_l^*$$

where Y_l are $p \times [n(t_l - t_{l-1})]$ matrices containing i.i.d. $\mathcal{N}(0,1)$ variables.

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High Dimensional Random Matrix Theory: Analyze the limiting spectral behavior of $[X]_p^n$:

For a diagonalisable $p \times p$ matrix A with real eigenvalues $\lambda_1, ..., \lambda_p$ the *spectral distribution* F^A is defined by the p.d.f.

$$F^A(x) = \frac{1}{p} \#\{i : \lambda_i \leq x\}.$$

In other words, $F^A(x)$ is the proportion of eigenvalues of A which are smaller or equal to x.

Section 2

Statement of the Main Result

theorem 2.1

Let $[X]_p^n = 1/n \sum_l T_l Y_l Y_l^* T_l^*$. *Assume*

(a) *that there is* $\tau_0 > 0$ *such that the largest eigenvalues of* $T_l T_l^*$ *are bounded by* τ_0 *for all* l*, uniformly in* p*.*

(b) *for all* $k > 0$ *and for all* $\mathbf{l} = (l_1, \ldots, l_k) \in \{1, \ldots, m\}^k$ *the existence of the mixed limiting spectral moments*

$$M_{\mathbf{l}}^k = \lim_{p \to \infty} \frac{1}{p} tr\left(\prod_{i=1}^{k} T_{l_i} T_{l_i}^* \right).$$

Then, the spectral distributions $F^{[X]_p^n}$ *converges weakly to a nonrandom p.d.f.* F*, a.s., which will be specified by its moment sequence. The moments* β_k *of* F *are of the form*

$$\beta_k = \sum_{r=1}^{k} c^{r-1} \sum_{\nu_1 + \ldots + \nu_r = k} \sum_{\mathbf{l}' \in \{1, \ldots, m\}^k} c_{r,\nu,\mathbf{l}'} \prod_{a=1}^{r} M_{\mathbf{l}(a)}^{\nu_a} \prod_{l=1}^{m} (t_l - t_{l-1})^{s_{l,\nu,\mathbf{l}'}}.$$

Section 3

Sketch of the Proof

The theorem extends the results of a seminal paper *(Y.Q. Yin and P.R. Krishnaiah, 1983)*, where the case $1/n\,TYY^*T^*$, i.e. $T_1 = \cdots = T_m$ was considered.

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- The authors worked with the assumption that TT^* has a limiting spectral distribution.
- In our framework, assuming the existence of LSDs for $T_1T_1^*, \ldots, T_mT_m^*$ is not sufficient.
- The existence of all mixed limiting spectral moments implies the existence of LSDs for $T_1T_1^*, \ldots, T_mT_m^*$.
- Additionally, it ensures a nice 'joint spectral behavior' of $T_1T_1^*, \ldots, T_mT_m^*$.

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Moment convergence theorem,

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Let (F_n) be a sequence of p.d.f.s with finite moments of all orders $\beta_{k,n} = \int x^k dF_n(x)$. Assume $\beta_{k,n} \to \beta_k$ for $n \to \infty$ for $k = 0, 1, \ldots$ where

(a) $\beta_k < \infty$ *for all* k *and*

(b) $\sum_{k=0}^{\infty} [\beta_{2k}(F)]^{-\frac{1}{2k}} = \infty$.

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Then, F_n converges weakly to the unique probability distribution function F with moment sequence $(\beta_k)_{k=0,\ldots}$.

Let $\lambda_1 \leq \ldots \leq \lambda_p$ be the eigenvalues of A, then

$$\beta_k(F^A) = \frac{1}{p} \sum_{i=1}^{p} \lambda_i^k = \frac{1}{p} \mathrm{tr}(A^k).$$

In order to determine $F = \lim F^{[X]_p^n}$ it is sufficient to find

$$\lim_{p,n\to\infty} \beta_k(F^{[X]_p^n}) = \lim_{p,n\to\infty} \frac{1}{p}\mathrm{tr}([X]_p^n)^k,$$

for $k = 1, 2, \ldots$.

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It holds that

$$\mathbb{E}[1/p\ \mathrm{tr}([X]_p^n)^k - \mathbb{E}[1/p\ \mathrm{tr}([X]_p^n)^k]]^4$$

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almost surely. Hence

$$\lim_{p,n\to\infty} \frac{1}{p}\mathrm{tr}([X]_p^n)^k = \lim_{p,n\to\infty} \frac{1}{p} \sum_{\mathbf{l}\in\{1,\dots,m\}^k} \mathbb{E}\left[\mathrm{tr}\left(\prod_{i=1}^k T_{l_i} Y_{l_i} Y_{l_i}^* T_{l_i}^*\right)\right],$$

where $\mathbf{l} = (l_1, \dots, l_k)$.

Expansion leads to

$$\mathbb{E}[\beta_k(F^{[X]_p^n})]$$

$$= p^{-1}n^{-k}\sum_{\mathbf{i},\mathbf{j},\mathbf{l}}\mathbb{E}\left[T_{l_1,i_1i_2}\underbrace{Y_{l_1,i_2j_1}}_{\sim\mathcal{N}(0,1)}Y^*_{l_1,j_1i_3}T^*_{l_1,i_3i_4}\cdots T_{l_k,i_{3k-2}i_{3k-1}}Y_{l_k,i_{3k-1}j_k}Y^*_{l_k,j_ki_{3k}}T^*_{l_k,i_{3k}i_1}\right]$$

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$\Rightarrow$ Combinatorical problem, solution uses graph theory.

We assign a colored graph to every summand

$$\mathbb{E}\left[T_{l_1,\, i_1 i_2} Y_{l_1, i_2 j_1} Y^*_{l_1, j_1 i_3} T^*_{l_1, i_3 i_4} \cdots T_{l_k, i_{3k-2} i_{3k-1}} Y_{l_k i_{3k-1} j_k} Y^*_{l_k, j_k i_{3k}} T^*_{l_k, i_{3k} i_1}\right].$$

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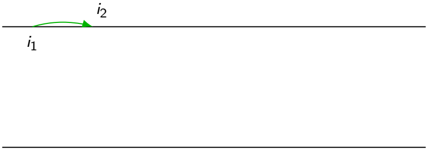

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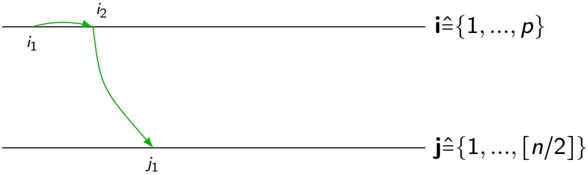

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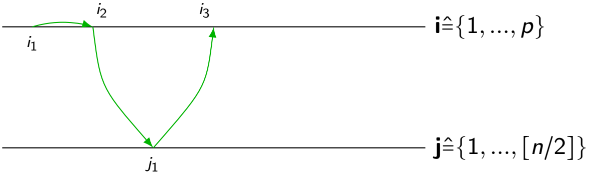

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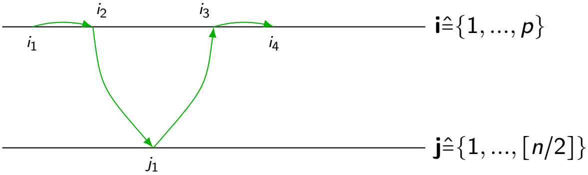

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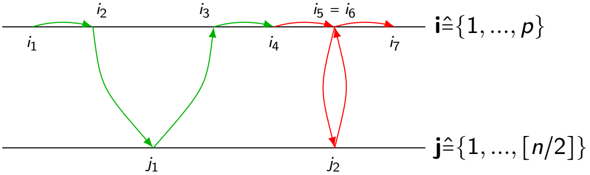

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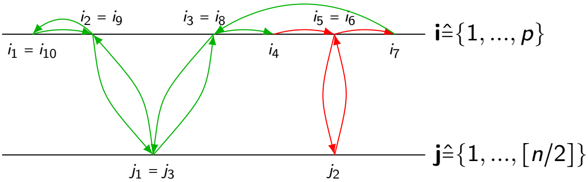

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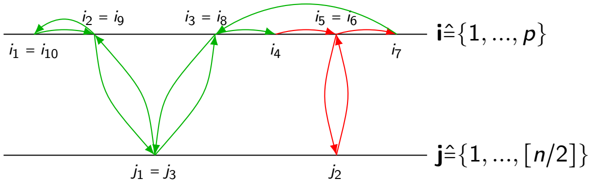
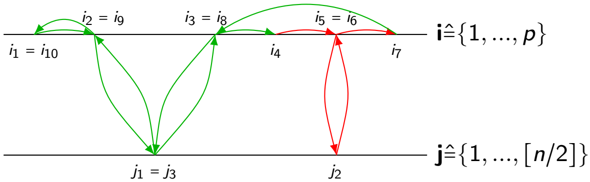


Advantage: One can divide the graphs into several categories according to their shape.

Example: Summing all summands

$$\mathbb{E}\left[T_{l_1,\,i_1 i_2} Y_{l_1,i_2 j_1} Y^*_{l_1,\,j_1 i_3} T^*_{l_1,i_3 i_4} \cdots T_{l_k,i_{3k-2} i_{3k-1}} Y_{l_k i_{3k-1} j_k} Y^*_{l_k,j_k i_{3k}} T^*_{l_k,i_{3k} i_1}\right]$$

corresponding to a graph with the shape



gives $\approx \mathrm{tr}(T_1 T_1^*)\mathrm{tr}(T_1 T_1^* T_2 T_2^*)$ if green $\hat{=}$ 1 and red $\hat{=}$ 2.

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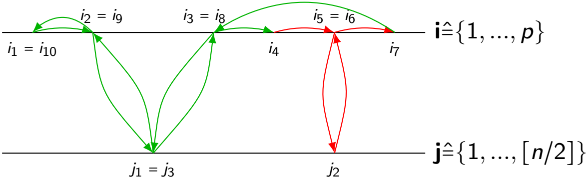

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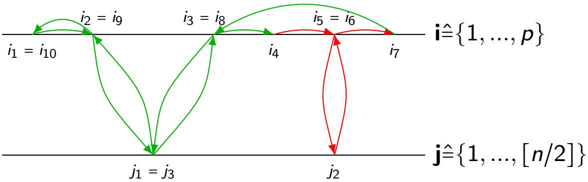
- Expectation factor $\mathbb{E}[Y_{l_1,i_2j_1}\cdots Y^*_{l_k,j_k i_{3k}}]$ is 1.

Example: Summing all summands

$$\mathbb{E}\left[T_{l_1,\, i_1 i_2} Y_{l_1, i_2 j_1} Y^*_{l_1, j_1 i_3} T^*_{l_1, i_3 i_4} \cdots T_{l_k, i_{3k-2} i_{3k-1}} Y_{l_k i_{3k-1} j_k} Y^*_{l_k, j_k i_{3k}} T^*_{l_k, i_{3k} i_1} \right]$$

corresponding to a graph with the shape



gives $\approx \mathrm{tr}(T_1 T_1^*)\mathrm{tr}(T_1 T_1^* T_2 T_2^*)$ if green $\hat{=}$ 1 and red $\hat{=}$ 2.

- Expectation factor $\mathbb{E}[Y_{l_1, i_2 j_1} \cdots Y^*_{l_k, j_k i_{3k}}]$ is 1.
- $1/p^2 \mathrm{tr}(T_1 T_1^*)\mathrm{tr}(T_1 T_1^* T_2 T_2^*)$ converges to $M^1_{(1)} M^2_{(1,2)}$.

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- For $T_1 = \ldots = T_m$ such a relation exists (Marčenko-Pastur equation), allowing the construction of consistent spectrum estimators (N. El Karoui, 2007)

Bibliography

Thank you for your attention!

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