Infill asymptotics for Lévy moving average processes

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Dynstoch 2014, September 11

Lévy moving average processes

- We consider a *Lévy moving average* process

$$X_t = X_0 + \int_{-\infty}^{t} g(t-s)dL_s,$$

where L is a pure jump Lévy process and the function g is assumed to be of the form

$$g(x) = x^\alpha f(x), \qquad \alpha > 0,$$

with $f : \mathbb{R}^+ \to \mathbb{R}$ being a smooth exponentially decaying function with $f(0) \neq 0$.

- Since $\alpha > 0$ the process X turns out to be continuous and stationary.

Power variations

- We define the *kth order differences of X* via

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}.$$

For instance,

$$\Delta_{i,1}^n X = X_{i/n} - X_{(i-1)/n} \quad \text{and} \quad \Delta_{i,2}^n X = X_{i/n} - 2X_{(i-1)/n} + X_{(i-2)/n}.$$

- The power variation of kth order differences of X is given by the statistic

$$V(X,p,k)_n := \sum_{i=k}^{n} |\Delta_{i,k}^n X|^p.$$

In the following we will study the asymptotic behaviour of the functional $V(X,p,k)_n$.

Background on Lévy processes

- Lévy motions are stochastic processes with *stationary* and *independent* increments, which are *continuous in probability*.

- A *pure jump Lévy process* L with Lévy measure ν and drift γ has the characteristic function

$$\mathbb{E}[\exp(iuL_t)] = \exp(t\psi(u))$$

with

$$\psi(u) = \gamma u + \int_{\mathbb{R}\setminus\{0\}} (\exp(iux) - 1 - iux1_{\{|x|\leq 1\}})\nu(dx).$$

Blumenthal-Getoor index

- Let $\Delta L_s = L_s - L_{s-}$ denote the jump of L at time s. The Blumenthal-Getoor index β is defined as

$$\beta := \inf\left\{r \geq 0 : \int_{-1}^{1} |x|^r \nu(dx) < \infty\right\}$$

$$= \inf\left\{r \geq 0 : \sum_{s\in[0,1]} |\Delta L_s|^r < \infty\right\}.$$

- For all Lévy processes it holds that

$$\sum_{s\in[0,1]} |\Delta L_s|^2 < \infty.$$

Hence, $\beta \in [0,2]$.

Symmetric β-stable Lévy processes

- A symmetric β-stable Lévy process (SβS) has a Lévy measure of the form

$$\nu(dx) = \text{const} \cdot |x|^{-1-\beta} dx, \qquad \beta \in (0,2).$$

Such a process is self-similar with index $1/\beta$, i.e.

$$(L_{at})_{t \geq 0} \overset{d}{=} (a^{1/\beta} L_t)_{t \geq 0}$$

- For β-stable Lévy processes it holds that

$$\beta = \text{Blumenthal-Getoor index.}$$

Assumptions on $X_t = X_0 + \int_{-\infty}^t g(t-s)dL_s$

Assumption (A):

(i) $g(x) = x^\alpha f(x)$ with $\alpha > 0$ and $f(0) \neq 0$.

(ii) For some $\theta > 0$ it holds that

$$\limsup_{t\to\infty} t^\theta \nu\{x : |x| > t\} < \infty$$

(iii) $g \in C^k(\mathbb{R}_{\geq 0})$,

$$|g^{(j)}(x)| \leq K|x|^{\alpha-j}, \qquad x \in (0,\delta)$$

and $g^{(j)} \in L^\theta((\delta,\infty))$ for some $\delta > 0$. Moreover, $|g^{(j)}|$ is decreasing on (δ,∞).

Remarks

- Assumption (A) guarantees the existence of the integrals

$$\int_{-\infty}^{t} g(t-s)dL_s \qquad \text{and} \qquad \int_{-\infty}^{t-\varepsilon} g^{(k)}(t-s)dL_s,$$

for any $\varepsilon > 0$, where k is the order of increments of X. The symmetry of L is not essential for most parts of the limit theory.

- We will see that the limit theory for power variation $V(X,p,k)_n = \sum_{i=k}^{n} |\Delta_{i,k}^{n} X|^p$ gives quite surprising results. In particular, it depends on the interplay between the parameters k, p, α and β.

- Only case (ii) below appeared in an earlier paper by Benassi, Cohen and Istas (04). However, their proof was incorrect.

First order asymptotics for $V(X,p,k)_n = \sum_{i=k}^n |\Delta_{i,k}^n X|^p$

Theorem: Assume that assumption (A) holds and L is a pure jump Lévy process with Blumenthal-Getoor index $\beta \in (0,2)$.

(i) If $\alpha \in (0, k - 1/p)$ and $p > \beta$, we obtain

$$n^{\alpha p} V(X,p,k)_n$$

$$\xrightarrow{d_{st}} |f(0)|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=k}^{\infty} |h_k(l + U_m)|^p \right),$$

where (T_m) are jump times of L, $(U_m)_{m \geq 1}$ is a sequence of iid $\mathcal{U}([0,1])$-distributed random variables and the function h_k is defined via

$$h_k(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)_+^{\alpha}.$$

First order asymptotics for power variation

Theorem (cont.):

(ii) Assume that L is a SβS process with $\beta \in (0,2)$. If $\alpha \in (0, k - 1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(X,p,k)_n \xrightarrow{\mathbb{P}} \mathbb{E}[|\widetilde{L}_1^{(k)}|^p]$$

where $\widetilde{L}^{(k)}$ is a SβS process defined via

$$\widetilde{L}_t^{(k)} := f(0) \int_{\mathbb{R}} h_k(t-s) dL_s.$$

When $k = 1$, $\widetilde{L}_t^{(1)} = f(0) \int_{\mathbb{R}} [(t-s)_+^{\alpha} - (t-s-1)_+^{\alpha}] dL_s$ is a *fractional β-stable Lévy noise.*

First order asymptotics for power variation

Theorem (cont.):

(iii) If $\alpha > k - 1/p,\ p > \beta$ or $\alpha > k - 1/\beta,\ p < \beta$, we deduce

$$n^{kp-1} V(X,p,k)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_s^{(k)}|^p ds$$

with

$$F_s^{(k)} = \int_{-\infty}^s g^{(k)}(s-u) dL_u.$$

Critical cases

Theorem (cont.):

(iv) If $\alpha = k - 1/p$ and $p > \beta$, we obtain

$$\frac{n^{\alpha p}}{\log n} V(X,p,k)_n \xrightarrow{\mathbb{P}} |f(0)|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p$$

(v) Assume that L is a SβS process with $\beta \in (0,2)$. If $\alpha = k - 1/\beta$ and $p < \beta/2$, we obtain

$$\frac{n^{p(\alpha+1/\beta)-1}}{(\log n)^{p/\beta}} V(X,p,k)_n \xrightarrow{\mathbb{P}} c_p^k$$

for a certain constant c_p^k.

Summary of first order asymptotics

Theorem:

(i) If $\alpha \in (0, k - 1/p)$ and $p > \beta$, we obtain

$$n^{\alpha p} V(X,p,k)_n \xrightarrow{d_{st}} |f(0)|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=k}^{\infty} |h_k(l + U_m)|^p \right),$$

(ii) Assume that L is a SβS process with $\beta \in (0,2)$. If $\alpha \in (0, k - 1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha + 1/\beta) - 1} V(X,p,k)_n \xrightarrow{\mathbb{P}} \mathbb{E}[|\widetilde{L}_1^{(k)}|^p]$$

(iii) If $\alpha > k - 1/p,\ p > \beta$ or $\alpha > k - 1/\beta,\ p < \beta$, we deduce

$$n^{kp-1} V(X,p,k)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_s^{(k)}|^p ds \quad \text{with} \quad F_s^{(k)} = \int_{-\infty}^{s} g^{(k)}(s-u) dL_u.$$

Remarks

- The rate of convergence in cases (i)-(iii) uniquely identifies the parameters α and β. This might be useful for statistical applications.
- Cases (ii) and (iii) can be extended to a functional convergence. Case (i) is more problematic.
- Also extensions to Lévy semi-stationary processes with non-trivial intermittency σ are in most cases relatively straightforward.
- Cases (i) and (iii) can be probably extended to general pure jump semimartingale drivers L.

Remarks

- Recall the convergence in case (i): If $\alpha \in (0, k - 1/p)$ and $p > \beta$, we obtain

$$n^{\alpha p} V(X,p,k)_n \xrightarrow{d_{st}} |f(0)|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=k}^{\infty} |h_k(l+U_m)|^p \right),$$

- The conditions $\alpha \in (0, k - 1/p)$ and $p > \beta$ are essentially sharp. Indeed, since $|h(x)| \le C x^{\alpha - k}$ for x large, we deduce that

$$\sum_{l=k}^{\infty} |h_k(l+U_m)|^p \le \text{const} < \infty \iff \alpha < k - 1/p.$$

  On the other hand, it holds that

$$\sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p < \infty,$$

  since β is the Blumenthal-Getoor index of L and $p > \beta$.

Sketch of proof: Case (ii)

- Assume for the moment that $k = 1$. We first show the approximation

$$X_{i/n} - X_{(i-1)/n} \approx f(0) \int_{\mathbb{R}} [(i/n - s)_+^\alpha - ((i-1)/n - s)_+^\alpha] dL_s$$
$$\stackrel{d}{=} n^{-(\alpha+1/\beta)} f(0) \int_{\mathbb{R}} [(i-s)_+^\alpha - (i-1-s)_+^\alpha] dL_s,$$

where the latter is a fractional β-stable Lévy noise, which is $(\alpha + 1/\beta)$-self similar. In a second step we apply the ergodic theorem.

- Another idea of proof, which usually requires existence of second moments (i.e. $p < \beta/2$), relies on the identity

$$|x|^p = a_p^{-1} \int_{\mathbb{R}} \frac{\exp(iux) - 1}{|u|^{1+p}} du, \qquad p \in (0,1),$$

and $a_p = \int_{\mathbb{R}} \frac{\exp(iu)-1}{|u|^{1+p}} du$. This identity connects the pth power with characteristic function.

Sketch of proof: Case (i)

- Assume for the moment that $k = 1$ and the Lévy process L has a single jump at time $T \in [0, 1]$. Let j be a random index such that

$$T \in [(j-1)/n, j/n)$$

- We first show the approximation

$$X_{i/n} - X_{(i-1)/n} \approx \int_{(i-1)/n}^{i/n} g(i/n - s) dL_s$$

$$+ \int_0^{(i-1)/n} [g(i/n - s) - g((i-1)/n - s)] dL_s$$

$$:= A_i^n + B_i^n$$

Sketch of proof: Case (i)

- Since $T \in [(j-1)/n, j/n)$ we have that

$$A_i^n \neq 0 \iff i = j,$$
$$B_i^n \neq 0 \iff i > j.$$

- Now, it holds that

$$|A_j^n|^p = \left| \int_{(j-1)/n}^{j/n} g(j/n - s) dL_s \right|^p = |\Delta L_T|^p |g(j/n - T)|^p$$
$$\approx |f(0)\Delta L_T|^p (j/n - T)^{\alpha p} \stackrel{d}{=} n^{-\alpha p} |f(0)\Delta L_T|^p U^{\alpha p},$$

where $U \sim \mathcal{U}([0,1])$. Similarly, it follows that $(l \geq 1)$

$$|B_{j+l}^n|^p \stackrel{d}{\approx} n^{-\alpha p} |f(0)\Delta L_T|^p \left((l+U)^\alpha - (l-1+U)^\alpha\right)^p.$$

Second order asymptotics associated with case (ii)

Theorem: Assume that L is a SβS process with $\beta \in (0,2)$.

(1) When $k \geq 2$, $\alpha \in (0, k-2/\beta)$ and $p < \beta/2$, we obtain

$$\sqrt{n}\Big(n^{p(\alpha+1/\beta)-1}V(X,p,k)_n - \mathbb{E}[|\widetilde{L}_1^{(k)}|^p]\Big) \Longrightarrow \mathcal{N}(0,v^2).$$

(2) When $k = 1$, $\alpha \in (0, 1-1/\beta)$ and $p < \beta/2$, it holds that

$$n^{1-\frac{1}{(1-\alpha)\beta}}\Big(n^{p(\alpha+1/\beta)-1}V(X,p,k)_n - \mathbb{E}[|\widetilde{L}_1^{(k)}|^p]\Big) \Longrightarrow S^{(1-\alpha)\beta},$$

where $S^{(1-\alpha)\beta}$ is a S$(1-\alpha)\beta$S random variable.

Remark: Part (2) uses the methods of Surgailis (04) established for discrete moving average processes.

Thank you!