Warwick Statistics

# Exact Simulation in a Nutshell

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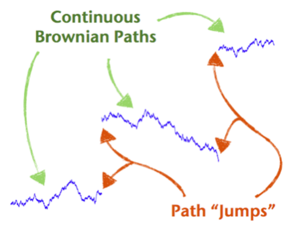
## 1 - Overview

**1.1 - The Goal. . . ???** Evaluate with certainty whether or not a given jump diffusion sample path crosses a barrier.

$$\mathbb{T}^x_{0,T}:\ \mathrm{d}X_t = \beta(X_{t-})\,\mathrm{d}t + \sigma(X_{t-})\,\mathrm{d}W_t + \mathrm{d}J_t^{\lambda,\mu}, \quad t\in[0,T],\ X_0 = x.$$

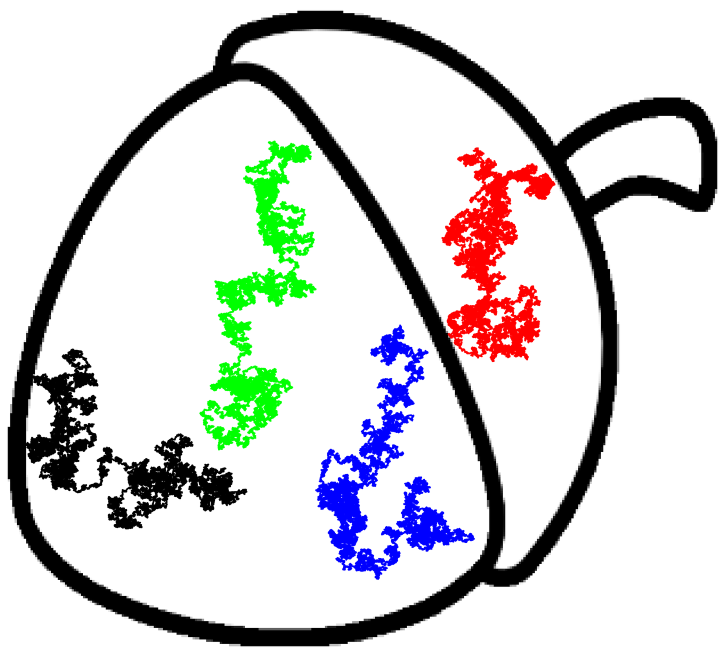
Target Law · Jump Diffusion · Instantaneous Volatility · Jump Intensity · Jump Size Density · Initial Value · Instantaneous Mean · Standard Brownian Motion · Compound Poisson Process · Time Interval




**1.2 - Main Difficulties. . . ???** Sample paths are infinite dimensional random variables. Discretisation schemes introduce error and don't sufficiently characterise sample paths to determine barrier crossing.

**1.3 - Applications. . .** Monte Carlo Integration, Option Pricing, Simulating First Hitting Times, Killed Diffusions, Rare Events. . .

## . . . in a nutshell. . .



## 2 - Summary of Key Methodology

**2.1 - Exact Algorithm (EA). . .** A diffusion path space retrospective rejection sampler which characterises entire (accepted) sample paths in the form of a finite dimensional **skeleton**, composed of the sample path at a finite collection of intermediate points and spatial information.

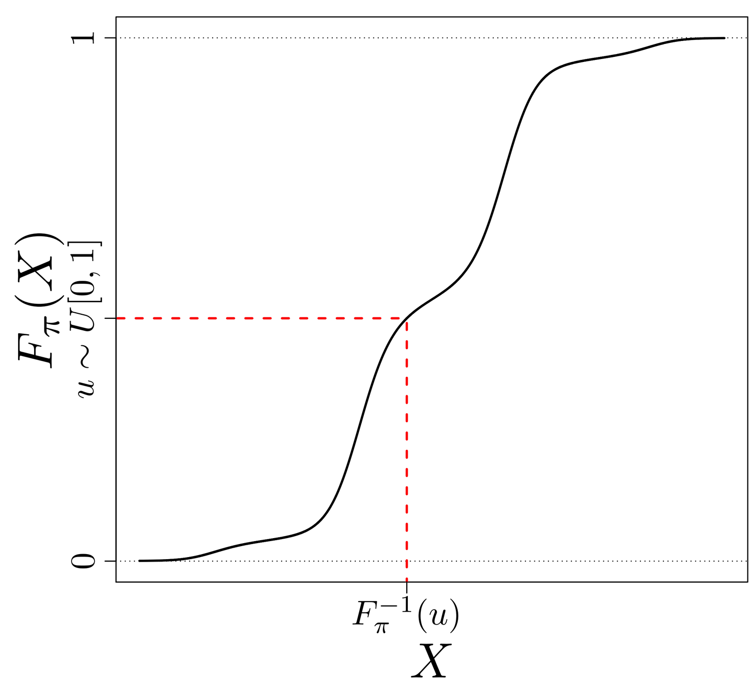
**2.2 - $\epsilon$-Strong Simulation ($\epsilon$SS). . .** Methodology for constructing upper and lower convergent dominating processes ($X^{\downarrow}$ and $X^{\uparrow}$), which enfold almost surely sample paths over some finite interval.

**2.3 - Sufficient Conditions. . .** $\beta\in C^1$, $\sigma\in C^2$ and strictly positive, $\lambda$ locally bounded, linear growth and Lipschitz continuity coefficient conditions.

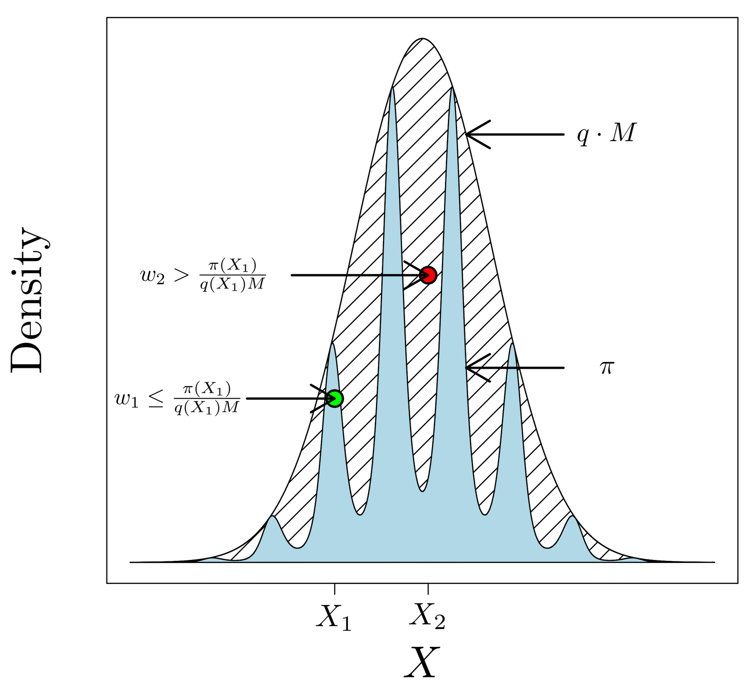
**2.4 - Further Details. . .** arXiv 1302.6964 or scan QR code!

## 3 - Key Ideas

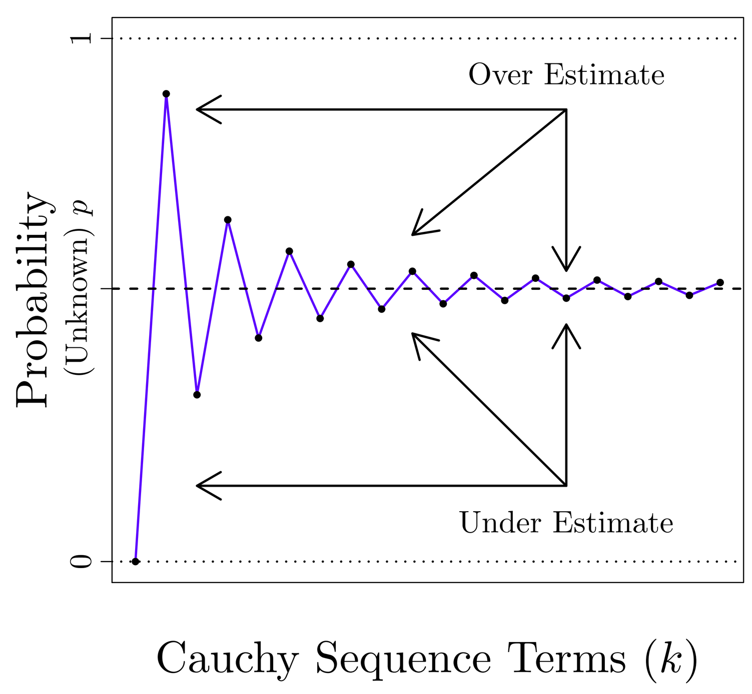
### Inversion Sampling



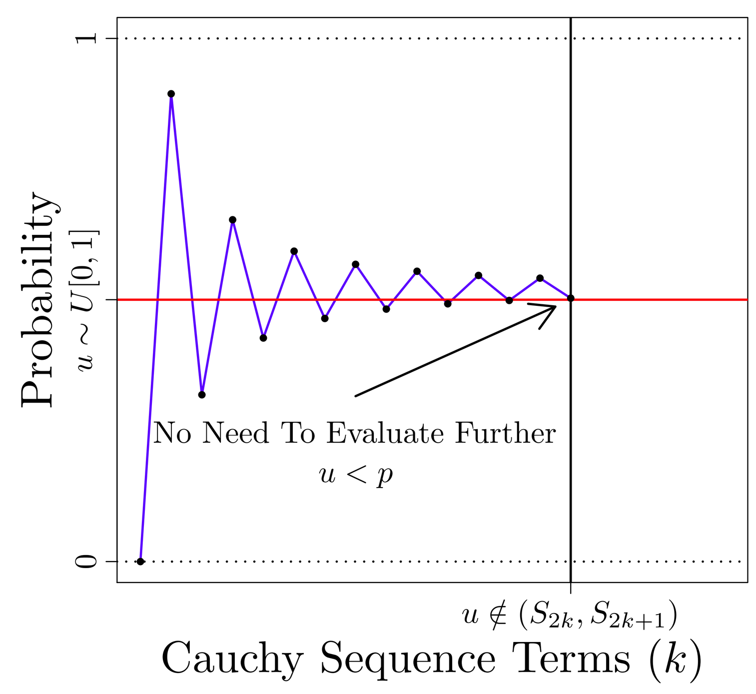

### Rejection Sampling




### Bernoulli Sampling




### Retro. Bernoulli Sampling




## 4 - The Exact Algorithm

**Idealised Exact Algorithm**

1 – Simulate $X\sim\mathbb{P}^x_{0,T}$.

2 – With probability $P_{\mathbb{P}^x_{0,T}}(X) := \frac{1}{M}\frac{\mathrm{d}\mathbb{T}^x_{0,T}}{\mathrm{d}\mathbb{P}^x_{0,T}}(X)\in[0,1]$ set $I=1$.

3 – $X|(I=1)\sim\mathbb{T}^x_{0,T}$.

**Key Points**

1 – $\mathbb{T}^x_{0,T}$ – target law.

2 – $\mathbb{P}^x_{0,T}$ – equivalent (+ *tractable*) proposal law.

3 – $\frac{\mathrm{d}\mathbb{T}^x_{0,T}}{\mathrm{d}\mathbb{P}^x_{0,T}}(X)$ bounded (by $M<\infty$).

**Key Idea:** Find and simulate some finite dimensional auxiliary random variable $F := F(X)\sim\mathbb{F}$, such that an unbiased estimator of the acceptance probability can be constructed which can be evaluated using only a finite dimensional subset of the proposal sample path. . .

**Implementable Exact Algorithm**

1 – Simulate $X_T := y\sim h$.

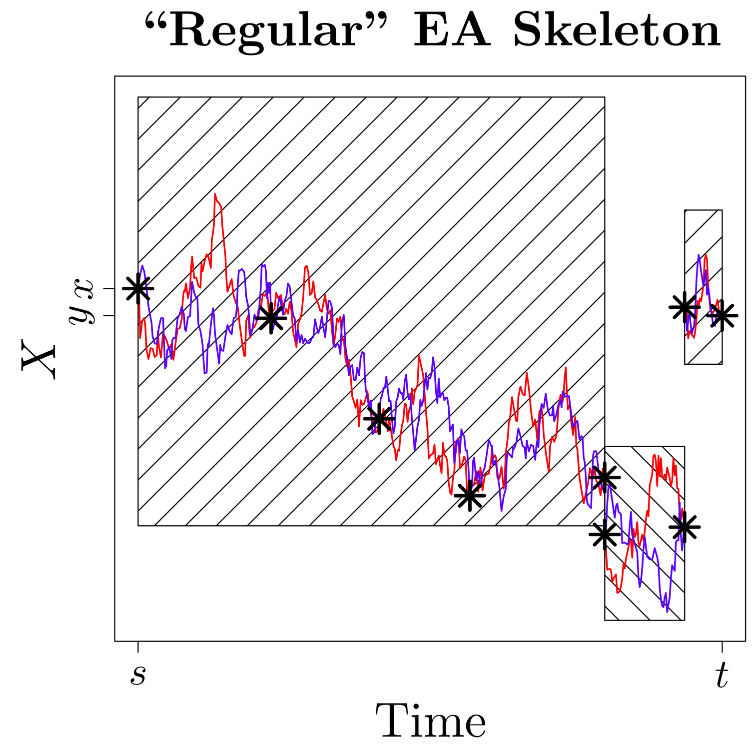
2 – Simulate $F\sim\mathbb{F}$.

3 – Simulate $X^f\sim\mathbb{P}^{x,y}_{0,T}\big|F$.

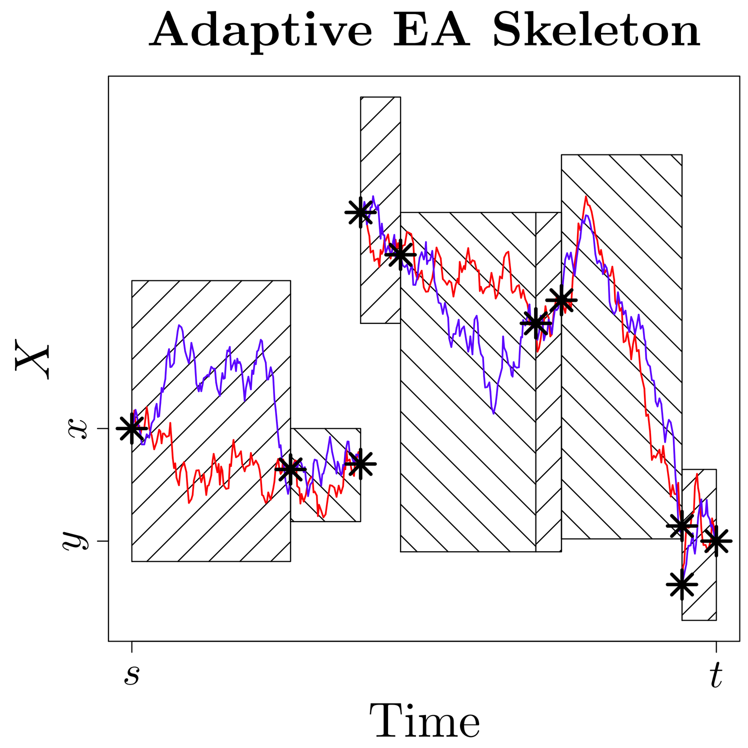
4 – With probability $P_{\mathbb{P}^x_{0,T}|F}(X)$ accept, else reject and return to 1.

5 – **** Simulate $X^c\sim\mathbb{P}^{x,y}_{0,T}\big|(X^f,F)$ as required. ****

### "Regular" EA Skeleton

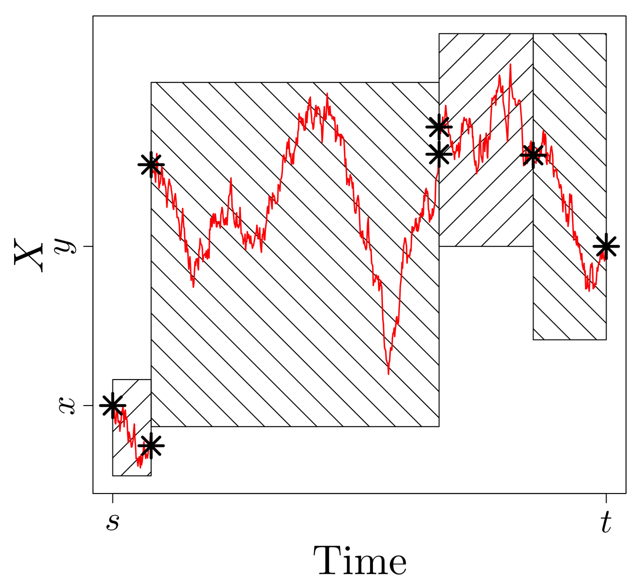



### Adaptive EA Skeleton

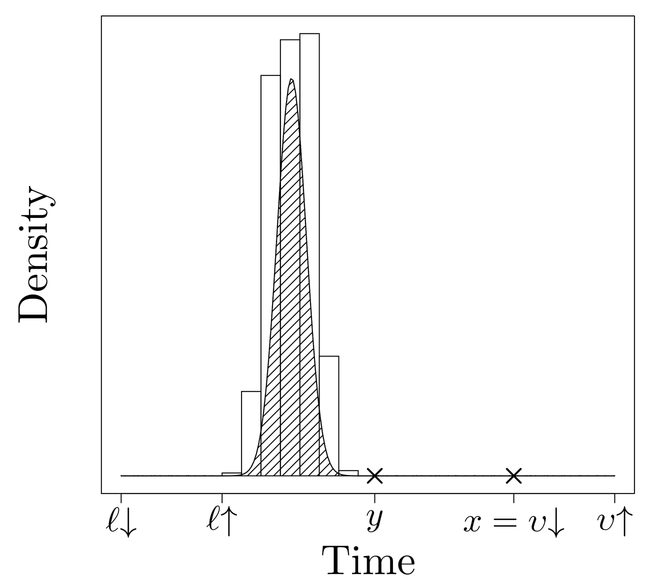



## 5 - $\epsilon$-Strong Simulation

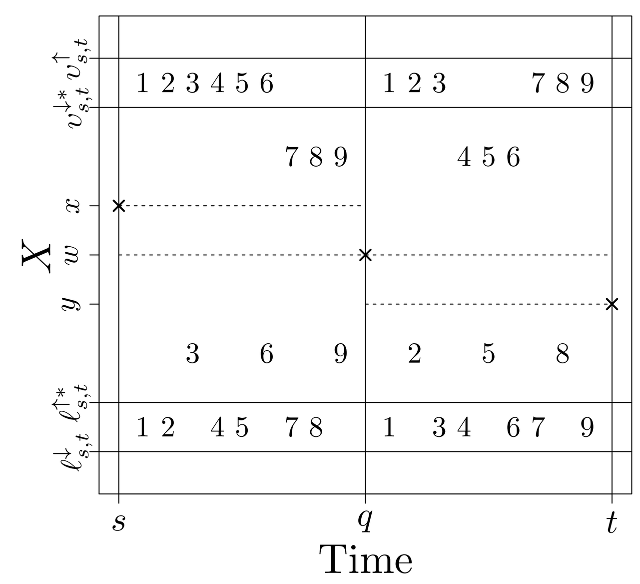
### Adaptive EA Skeleton



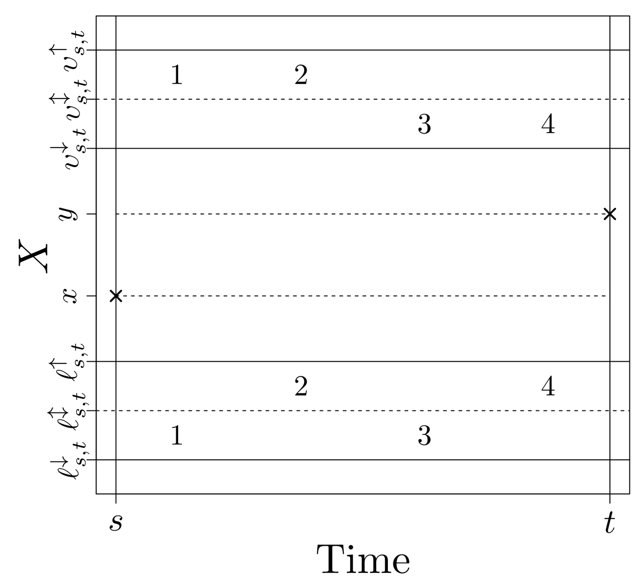

### Intermediate Augmentation



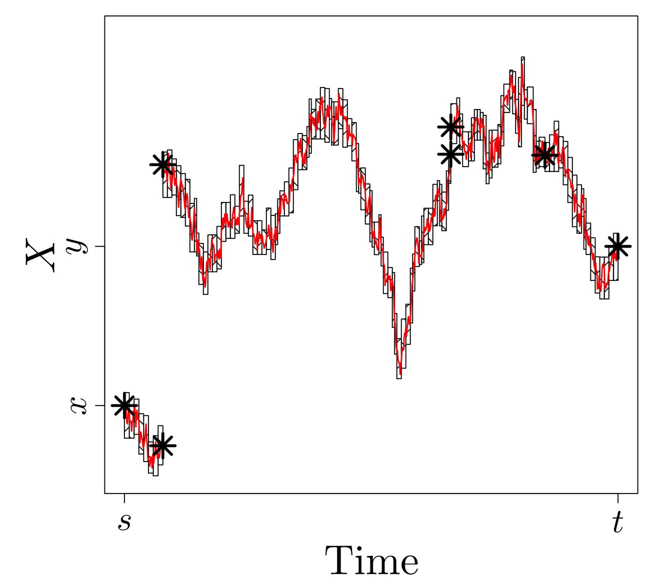

### Layer Dissection




### Layer Refinement

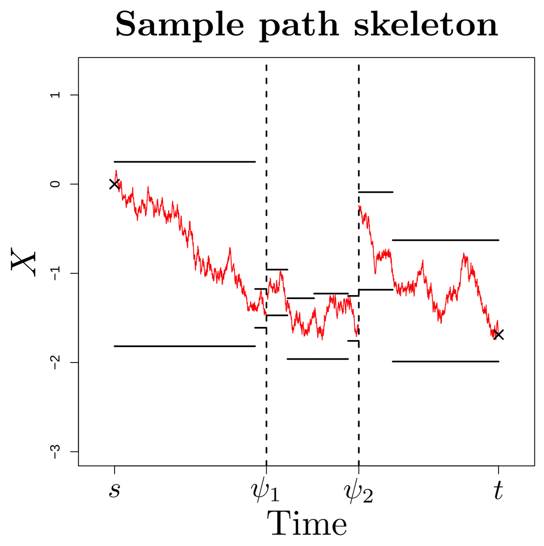



### Augmented Skeleton

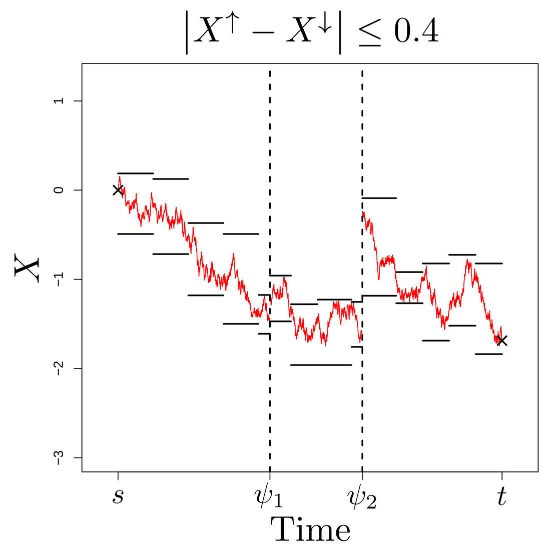



## 6.1 - Example 1: $\epsilon$-Strong Simulation of Jump Diffusions

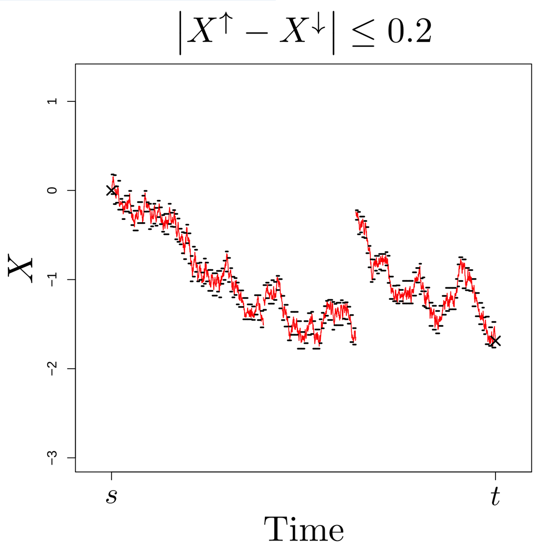
### Sample path skeleton




### $|X^{\uparrow}-X^{\downarrow}|\le 0.4$

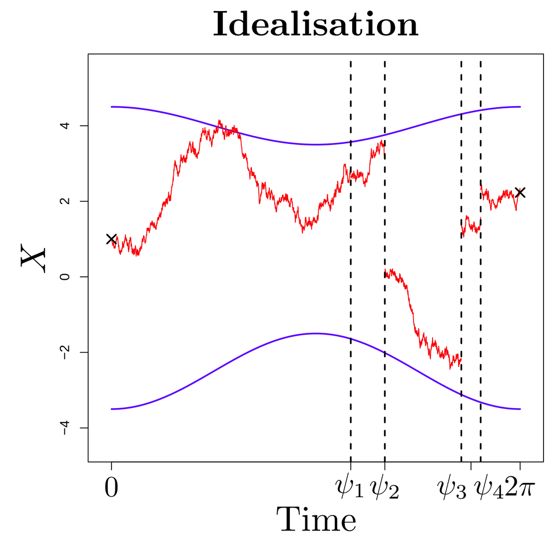



### $|X^{\uparrow}-X^{\downarrow}|\le 0.2$

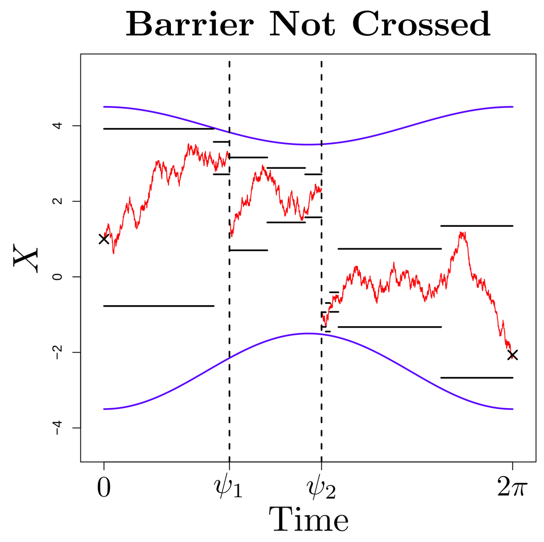



## 6.2 - Example 2: Nonlinear Two Sided Barrier

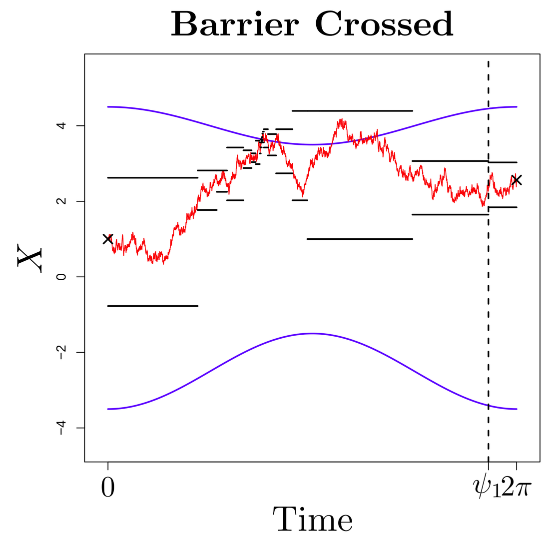
### Idealisation




### Barrier Not Crossed

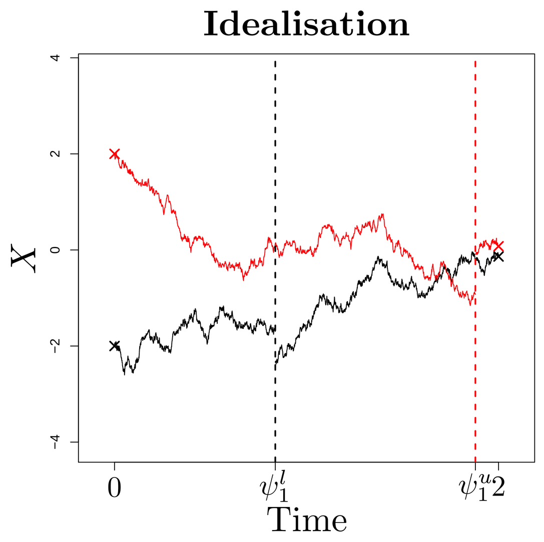



### Barrier Crossed

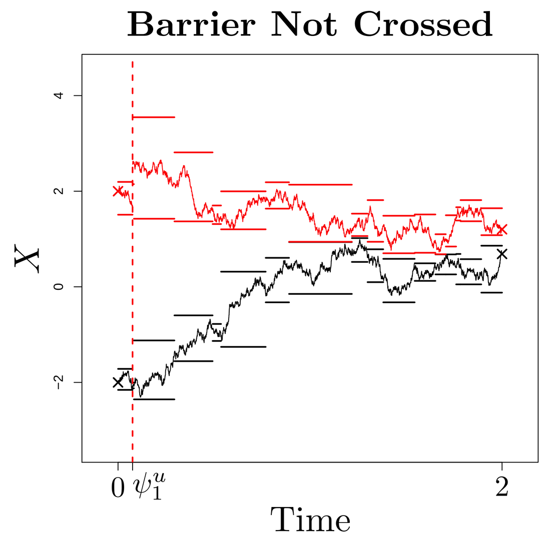



## 6.3 - Example 3: Jump Diffusion Intersection

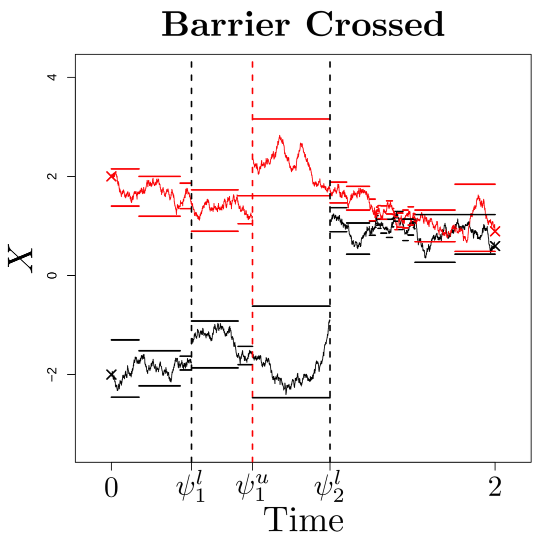
### Idealisation




### Barrier Not Crossed

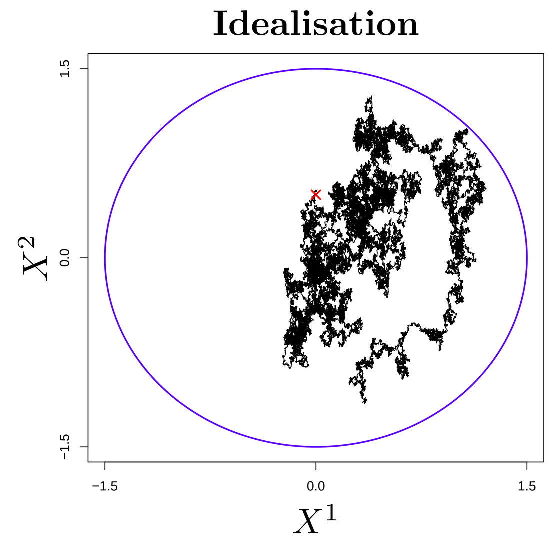



### Barrier Crossed

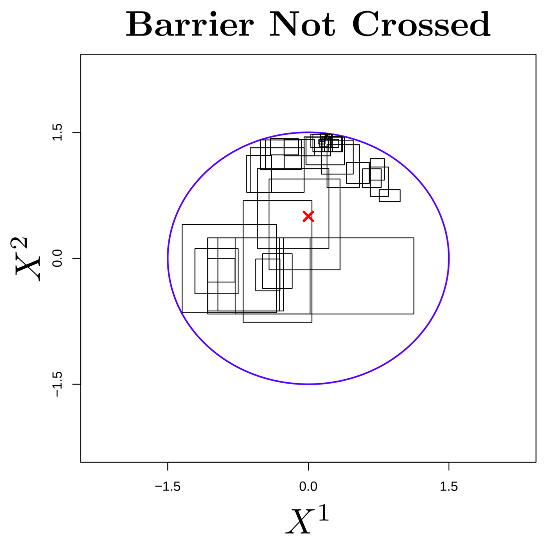



## 6.4 - Example 4: Circular Barrier

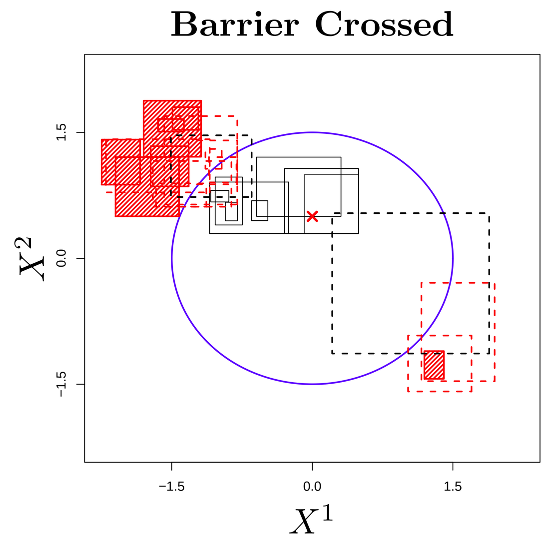
### Idealisation




### Barrier Not Crossed




### Barrier Crossed




*"O God, I could be bounded in a nut shell and count myself a king of infinite space, were it not that I have bad dreams,"* — William Shakespeare, *Hamlet Act II, Scene II*

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