Stochastic Spikes and Poisson Approximation of one-dimensional SDEs with applications to continuously measured Quantum Systems

– joint work with Martin Kolb –

Motivation

The following SDE is of interest for quantum mechanic models:

$$dX_t = \frac{\lambda^2}{2}(\varepsilon - b \cdot X_t)\, dt + \lambda X_t\, dB_t, \qquad X_0 = x > 0$$

with $b, \lambda, \varepsilon > 0$. $\lambda \to \infty$ acts as time acceleration and by $\varepsilon \to 0$ the positive drift is reduced.

Simulations

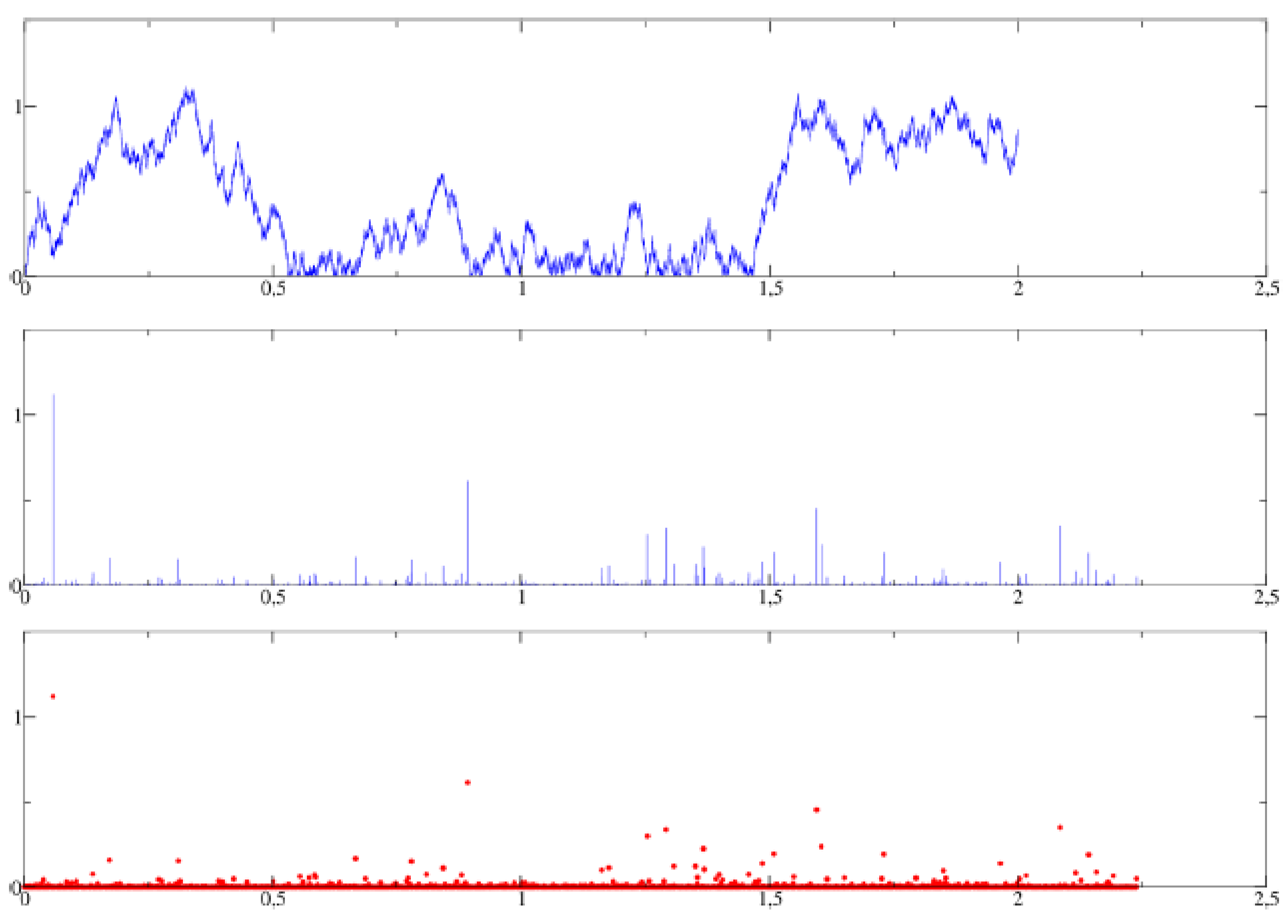


Figure: Scaling limit to Poisson process. Graphic taken from first reference.

Result by M. Bauer and D. Bernard

In scaling limit $\lambda \to \infty, \varepsilon \to 0, \lambda^2\varepsilon^{b+1} = \mathfrak{J}$ hitting time T_z has law:

$$T_z \overset{\mathbb{P}_x}{\sim} \left(\frac{x}{z}\right)^{b+1} \delta_0 + \left(1 - \left(\frac{x}{z}\right)^{b+1}\right) \mathrm{Exp}_{\frac{\mathfrak{J}(b+1)}{2\Gamma(b+1)} \frac{1}{z^{b+1}}}.$$

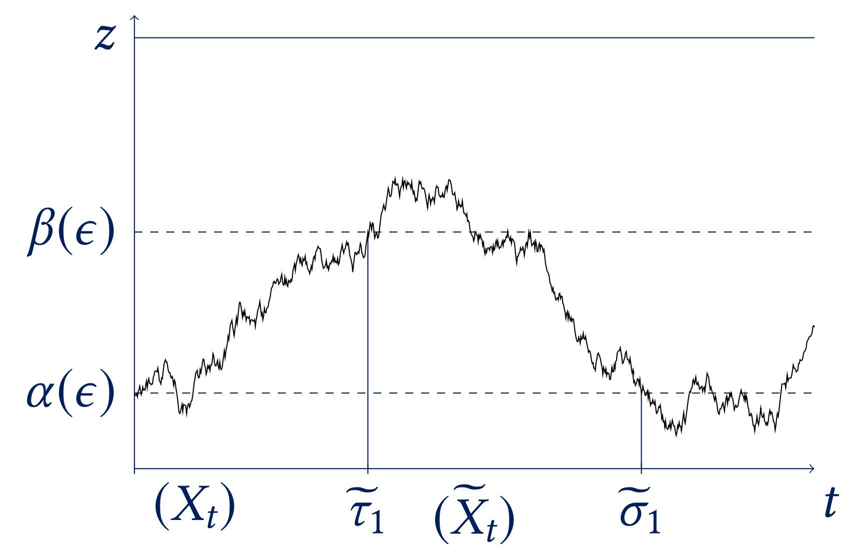
Conjecture by M. Bauer and D. Bernard

Appropriately scaled the corresponding assertion holds for a larger class of SDEs with form

$$dX_t = \frac{\lambda^2}{2}(\varepsilon \cdot b_1(X_t) - b_2(X_t))\, dt + \lambda \cdot \sigma(X_t)\, dB_t.$$

Conceptual idea

Cycle decomposition:



Assumptions

(A1) The given SDE allows for a (weak) solution unique in law.

(A2) $$\mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1] \xrightarrow[\varepsilon \downarrow 0]{} \kappa^{-1} \in (0, \infty).$$

(A3) $$\limsup_{\varepsilon \downarrow 0} \mathbb{E}_{\alpha(\varepsilon)}[\widetilde{\sigma}_1^2] < \infty.$$

(B1) For all values $z > 0$ and starting points $0 < x < z$

$$T_{\alpha(\varepsilon)} \wedge T_z \xrightarrow[\text{scaling}]{\mathcal{D}} 0,$$

and the law of $T_{\alpha(\varepsilon)}$ under $\mathbb{P}_x(\cdot \mid T_{\alpha(\varepsilon)} < T_z)$ converges to δ_0 and

(B2) $$\mathbb{P}_x(T_{\alpha(\varepsilon)} < T_z) \xrightarrow[\varepsilon \to 0]{} \alpha_{x,z} \in (0, 1).$$

Main result

If all assumptions (A1)–(A3), (B1) and (B2) are met, the law of the hitting time T_z when started at $0 < x < z$ converges in the generalized scaling limit $\lambda \to \infty, \varepsilon \to 0, \lambda^2 \mathbb{P}_{\beta(\varepsilon)}(T_z < T_{\alpha(\varepsilon)}) = \mathfrak{J}$ to

$$(1 - \alpha_{x,z})\delta_0 + \alpha_{x,z}\, \mathrm{Exp}_{\mathfrak{J}\kappa}.$$

Asymptotic linear SDEs

Consider the class of coefficient functions given by

(E1) b_1 is positive and continuously differentiable with

$$\inf b_1 > 0 \text{ and } \sup b_1 < \infty.$$

(E2) b_2 is nonnegative twice continuously differentiable with

$$b_2(0) = 0 \text{ and } b_2'(0) > 0.$$

(E3) σ is twice continuously differentiable with

$$\sigma(x) = 0 \Leftrightarrow x = 0 \text{ and } \sigma'(0) > 0.$$

Homodyne detection of Rabi oscillation

Of independent interest is the case

$$b_1(x) = 1, \quad b_2(x) = b \cdot x \ \ (b > 0), \quad \sigma(x) = x^2.$$

Selected references

- M. Bauer and D. Bernard, Stochastic spikes and strong noise limits of stochastic differential equations, Annales Henri Poincaré, 19(3), 2018, 653–693
- M. Kolb and M. Liesenfeld, Stochastic spikes and Poisson Approximation of one-dimensional stochastic differential equations with applications to continuously measured Quantum Systems, 2018, arXiv:1804.09501

Matthias Liesenfeld
liesenfe@math.uni-paderborn.de
Warburger Str. 100, D-33098 Paderborn

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