Constructing Bernoulli factories via perfect sampling of Markov chains

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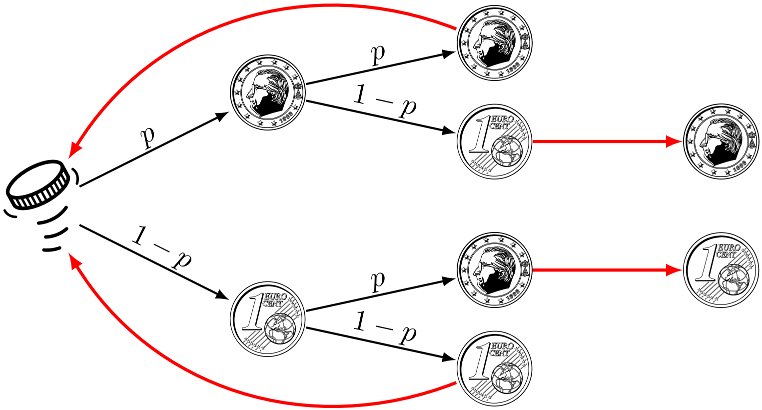
A fair game out of a biased coin

[von Neumann, 1951]

Given an unfair coin, how can you produce fair results?



Bernoulli Factory

Bernoulli Factory

Given a p-coin, i.e. a Bernoulli random variable with **unknown** mean $p \in \mathcal{D}$, with $\mathcal{D} \subseteq (0,1)$, and a **known** function $f : \mathcal{D} \to (0,1)$, for which functions f is it possible to simulate a $f(p)$-coin? (and how?)

Some possible functions:

- $f(p) = \frac{1}{2}, \qquad p \in (0,1)$
- $f(p) = c, \qquad p, c \in (0,1)$
- $f(p) = p^2, \qquad p \in (0,1)$
- $f(p) = \sqrt{p}, \qquad p \in (0,1)$
- $f(p) = 2p, \qquad p \in \left(0, \frac{1}{2}\right)$
- $f(p) = 2p, \qquad p \in \left(0, \frac{1}{2} - \epsilon\right)$

Existence of a Bernoulli Factory

[Keane and O'Brien, 1994]

Let $\mathcal{D} \subseteq (0,1)$ and let $f : \mathcal{D} \to (0,1)$. The function f is simulable **if and only if**

1. f is continuous on $\mathcal{D}$, and
2. either f is constant on $\mathcal{D}$ or $\forall p \in \mathcal{D}$ there exists an $n \in \mathbb{N}$ such that

$$\min(f(p), 1 - f(p)) \geq \min(p^n, (1-p)^n)$$

Application of Bernoulli Factories

Bernoulli Factories have been successfully applied to:

- Further understanding of black box algorithms for **processing randomness** [Flajolet et al., 2011];
- **Intractable likelihood** problems in Bayesian inference [Gonçalves et al., 2017];
- **Exact simulation** of diffusions [Łatuszyński et al., 2011];
- **Perfect simulation** on uncountable support [Flegal and Herbei, 2012];
- **Mechanism design** [Dughmi et al., 2017];
- Provable **quantum** advantage [Dale et al., 2015].

Main contributions

Main objectives of this work

- Extend **from coins to dice**;
- Provide a **novel way** to tackle this kind of problems;
- Construct an algorithm to target **rational functions**.

Dice Enterprise for rational functions

Given a rational function $f : \Delta^m \to \Delta^v$

$$f(\boldsymbol{p}) = (f_1(\boldsymbol{p}), \dots, f_v(\boldsymbol{p})) = \frac{1}{C(\boldsymbol{p})}(G_1(\boldsymbol{p}), \dots, G_v(\boldsymbol{p}))$$

how can we get a sample from the v-sided die associated to $f(\boldsymbol{p})$ by just rolling an m-sided die where the probability of rolling each face is given by $\boldsymbol{p}$?

General idea of the method

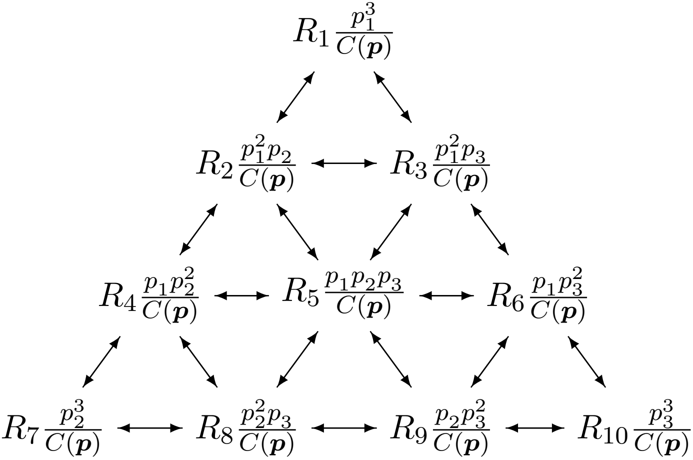
Given a rational function $f(\boldsymbol{p}) = \frac{1}{C(\boldsymbol{p})}(G_1(\boldsymbol{p}), \dots, G_v(\boldsymbol{p}))$

1. **Decomposition**
   Rewrite it as single monomials of the same degree

$$\pi(\boldsymbol{p}) = \frac{1}{C(\boldsymbol{p})} \left(R_1 \prod_{j=1}^{m} p_j^{n_{1,j}}, R_2 \prod_{j=1}^{m} p_j^{n_{2,j}}, \dots, R_k \prod_{j=1}^{m} p_j^{n_{k,j}} \right)$$

   so that sampling from $\pi(\boldsymbol{p})$ is equivalent to sampling from $f(\boldsymbol{p})$.

2. **Construct a Markov chain that admits $\pi(\boldsymbol{p})$ as stationary distribution**



General idea of the method

3. **Sample from the stationary distribution of the chain using perfect simulation**
   We can use Coupling From the Past to get a sample from $\pi(\boldsymbol{p})$. In fact, we can simulate from the chain **by just rolling the given** m**-sided die**.

General idea of the method

3. **Sample from the stationary distribution of the chain using perfect simulation**
   We can use Coupling From the Past to get a sample from $\pi(\boldsymbol{p})$. In fact, we can simulate from the chain **by just rolling the given m-sided die**.

Thank you!

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