The ZigZag Process

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In the last few years there is a tendency to use non-reversible Markov Processes to simulate from the invariant distribution.

A new class of processes used is Piecewise Deterministic Markov Processes.

They follow deterministic dynamics for some random period of time. Then they switch to different dynamics.

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The Zig-Zag process in $\mathbb{R}$

In 1-dimension, the process moves with constant speed $+1$ to the right and then changes to the left etc.

The state space is $\mathbb{R} \times \{-1, +1\}$

The rardomness if formulated in terms of a Poisson Process which is charecterized by a function $\lambda(x, \theta)$.

If we start from $(0, +1)$, continue in this direction until time T_1. T_1 is the first event of a Poisson Process with intensity $\{\lambda(t, +1), t \geq 0\}$.

If we change at T_1 continue towards the left and change again at time $T_1 + T_2$ where T_2 is the first point of a Poisson Process with intensity $\{\lambda(T_1 - t), \tau \geq 0\}$

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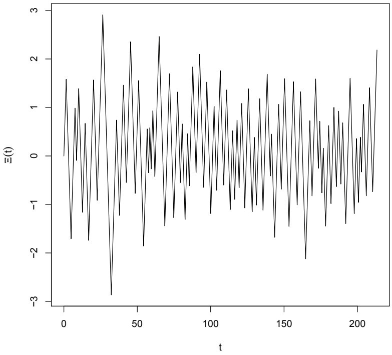

Figure: Normal Distribution
Picture by J. Bierkens

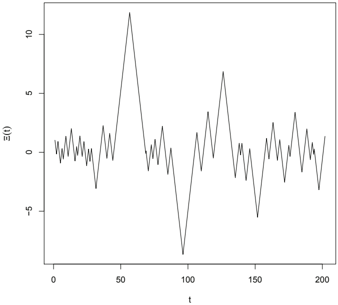

Figure: Cauchy Distribution
Picture by J. Bierkens

Invariant Measure

All randomness is hidden in λ.

Proposition

If $\pi(dx, d\theta) = \frac{1}{Z}\exp\{-U(x)\}(dx, d\theta)$

then the Zig-Zag process with rates

$$\lambda(x,\theta) = (\theta U'(x))^{+} + \gamma(x)\ ,\ \gamma \geq 0$$

has π *as unique invariant measure.*

Theorem (Bierkens-Roberts-Zitt (2017))

Consider (X_t, Θ_t) *in* $\mathbb{R}^d$*. If* $U \in C^3$*, there exist* $c > d, c'$ *with* $U(x) \geq clog(1 + |x|) - c'$ *and* U *has a non-degenerate local minimum, then*

$$\lim_{t\to\infty} \sup_A ||\mathbb{P}_{x,\theta}[(X_t, \Theta) \in A] - \pi(A)||_{TV} = 0.$$

This implies that for all $f \in L^1(\pi)$*,*

$$\frac{1}{T}\int_0^T f(X_t)dt \to \mathbb{E}_\pi[f(X)] \text{ as } T \to \infty \text{ a.s.}$$

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Central Limit Theorem

Theorem (Bierkens-Roberts-Zitt (2017))

Under some heavier conditions on the growth of U and assuming that it has lighter tails, we also have a CLT for the Zig-Zag

$$\frac{1}{\sqrt{n}} \int_0^n f(X_s, \Theta_s) - \pi(f) ds \to \mathcal{N}(0, \sigma_f^2)$$

in distribution as $n \to \infty$, for some $0 \leq \sigma_f < \infty$.

My research

Why restrict to $\{-1, +1\}^d$ velocities?

Allow $\{-\theta_m, ..., -\theta_1, \theta_1, .., \theta_m\}^d$.

Need more Poisson Processes, for any pair of possible speeds θ_i, θ_j need a function $\lambda(x, \theta_i, \theta_j)$.

More freedom to choose the algorithm. However seems to be heavier computationally.

When we only allow jumps to "neighbouring" speeds, the Ergodicity Theorem seems to hold, using basically the same arguments.