Exact Simulation of the Supremum of a Stable Process

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Talk for *LMS Invited Lecture Series and CRISM Summer School in Computational Statistics 2018*

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The Problem

- Simulation of the supremum of Lévy processes is a tough problem:
  - Few cases with exact simulation in the infinite activity case (even worse for the infinite variation case!)
  - It is not known how good discretisations are.

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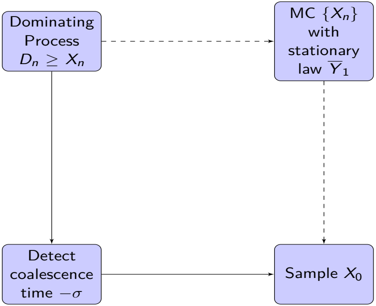
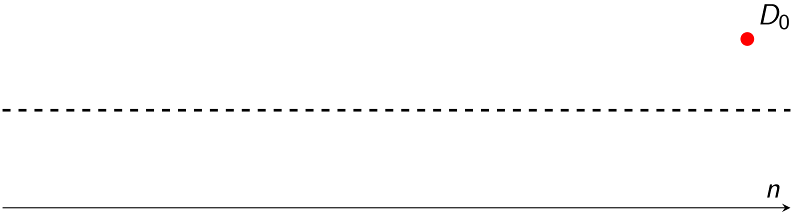
  - Few cases with exact simulation in the infinite activity case (even worse for the infinite variation case!)
  - It is not known how good discretisations are.
- **Stable** processes:
  - Often used as classical examples because their self-similarity often allow for **closed form** formulas.
  - Even here, only spectrally one-sided cases seem feasible from the literature [BDP11]

The Ingredients and the Strategy



The Main Idea



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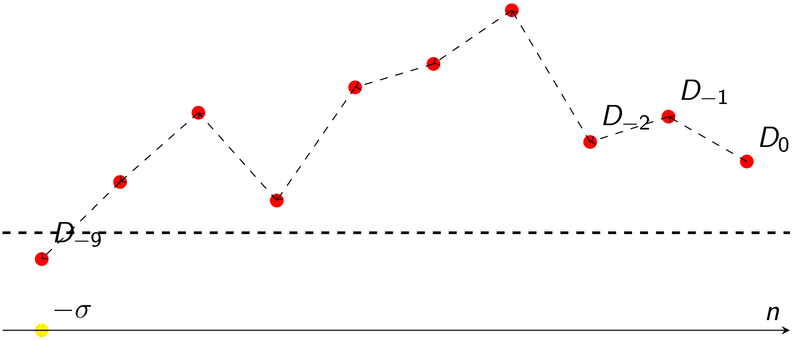

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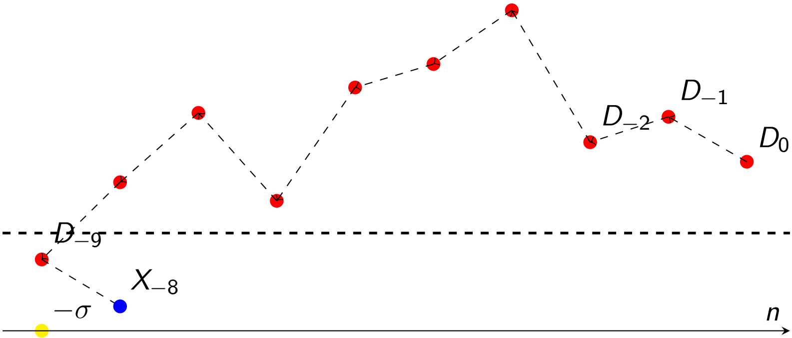

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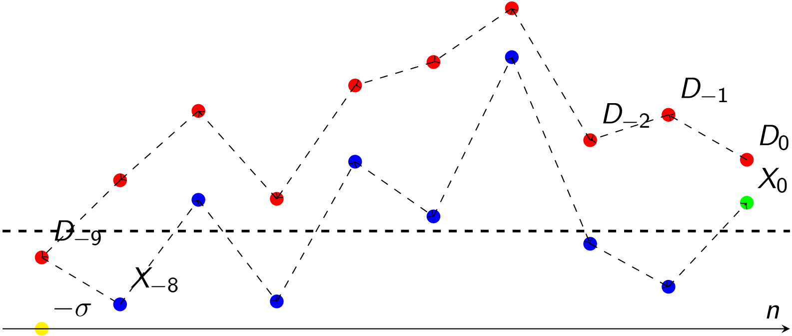

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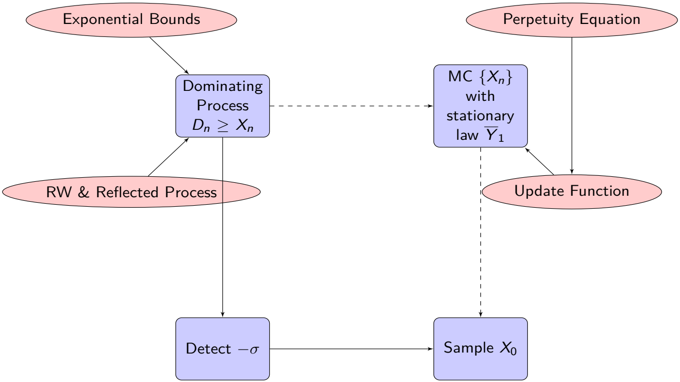


Outline of the Talk

Stochastic Perpetuity

Markov Chain

Dominating Process

Stable Processes

Using Zolotarev's (C) form, given any $\alpha \in (0, 2]$ and any skewness parameter $\beta \in [-1, 1]$

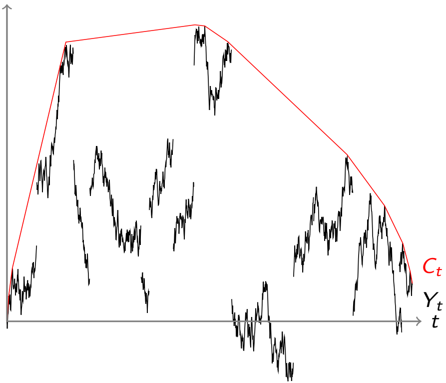
$$\rho = \mathbb{P}(Y_1 > 0) = \frac{\theta + 1}{2}, \quad \theta = \beta \left(\frac{\alpha - 2}{\alpha} 1_{\alpha > 1} + 1_{\alpha \leq 1} \right),$$

then

$$\log \left(\mathbb{E} \left(e^{itY_1} \right) \right) = - |t|^{\alpha} e^{-i \frac{\pi \alpha}{2} \theta \operatorname{sgn}(t)}.$$

Concave Majorant

Fix a Lévy process $\{Y_t\}$. Its concave majorant is the (random) smallest concave function $\{C_t\}$ that dominates $\{Y_t\}$.



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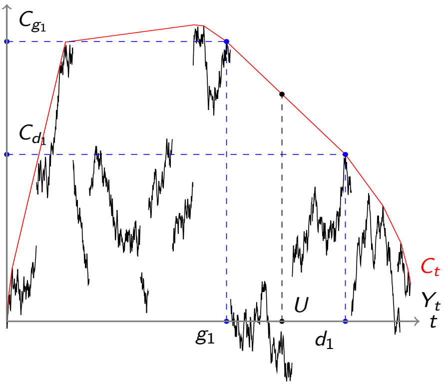


Concave Majorant

Discover the faces of C independently at random, uniformly on lengths. Then the faces satisfy [PUB12]:

$$\{(d_n - g_n, C_{d_n} - C_{g_n})\}_n \overset{d}{=} \{(\ell_n, Y_{L_n} - Y_{L_{n-1}})\}_n \overset{d}{=} \left\{\left(\ell_n, \ell_n^{\frac{1}{\alpha}} Z_n\right)\right\}_n,$$

for independent iid $U_n \sim U(0,1)$, $\ell_n = U_n(1 - L_{n-1})$, and $L_n = \sum_{i=1}^{n-1} \ell_i$ (**stick-breaking process**) and an independent iid sequence $Z_n \overset{d}{=} Y_1$. Then $\overline{Y}_1 := \sup_{t\in[0,1]} Y_t$ satisfies

$$\begin{aligned} \overline{Y}_1 &= \sum_{n=1}^{\infty} \ell_n^{\frac{1}{\alpha}} Z_n^+ = \ell_1^{\frac{1}{\alpha}} Z_1^+ + (1-\ell_1)^{\frac{1}{\alpha}} \sum_{n=2}^{\infty} \left(\frac{\ell_n}{1-\ell_1}\right)^{\frac{1}{\alpha}} Z_n^+ \\ &\overset{d}{=} U^{\frac{1}{\alpha}} \overline{Y}_1 + (1-U)^{\frac{1}{\alpha}} Z_1^+. \end{aligned}$$

Stochastic Perpetuity

Let $S^+(\alpha,\rho)$ and $\overline{S}(\alpha,\rho)$ be the laws of Y_1 conditioned on being positive and of $\overline{Y}_1$ respectively. Then, the relation for the faces of C and the scaling property of stable processes then yield [GCMUB18]:

$$\overline{\mathbf{Y}}_{\mathbf{1}} \stackrel{d}{=} \left(1 + B\left(V^{\frac{1}{\alpha\rho}} - 1\right)\right)\left(U^{\frac{1}{\alpha}}\overline{\mathbf{Y}}_{\mathbf{1}} + (1-U)^{\frac{1}{\alpha}} S\right),$$

where $(B, U, V, S, \zeta) \sim Ber(\rho) \times U(0,1)^2 \times S^+(\alpha,\rho) \times \overline{S}(\alpha,\rho)$. And $\overline{S}(\alpha,\rho)$ is the unique solution.

First Update Function

- Let $\Theta = (U, W, \Lambda, S)$ for an independent $W \sim U(0,1)$. Then the perpetuity may be summarised as

$$\overline{Y}_1 \stackrel{d}{=} \phi\left(\overline{Y}_1, \Theta\right),$$

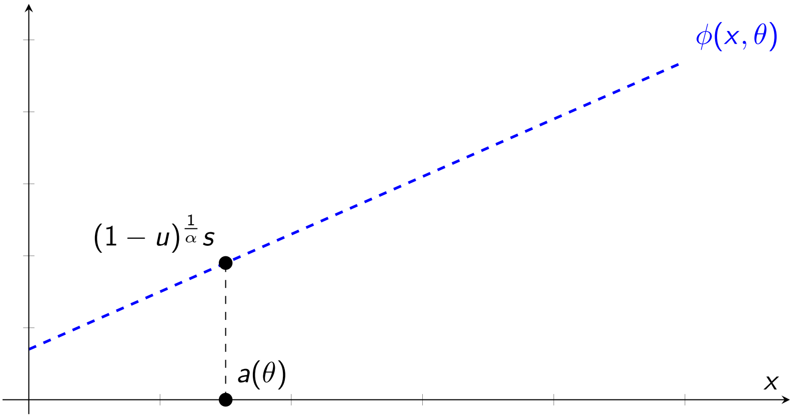
where $\Lambda = 1 + B\left(V^{1/\rho} - 1\right)$ and

$$\phi(x, \theta) = \lambda^{\frac{1}{\alpha}}\left(u^{\frac{1}{\alpha}} x + (1-u)^{\frac{1}{\alpha}} s\right).$$

- Consider the functions

$$\begin{aligned} a(\theta) &= \left(\lambda^{-\frac{1}{\alpha}} - 1\right) u^{-\frac{1}{\alpha}} (1-u)^{\frac{1}{\alpha}} s, \\ \psi(x, \theta) &= 1_{\{x \le a(\theta)\}} w^{\frac{1}{\alpha\rho}} (1-u)^{\frac{1}{\alpha}} s + 1_{\{x > a(\theta)\}} \phi(x, \theta). \end{aligned}$$

Update Functions



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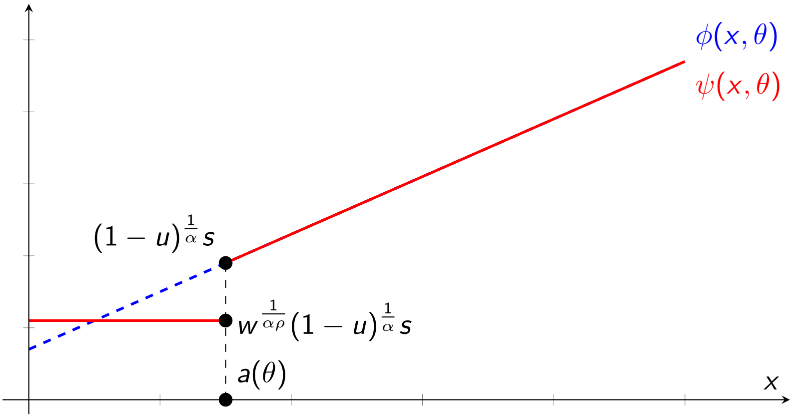


Second Update Function

- Then $X \sim \overline{S}(\alpha, \rho)$ is the unique solution to

$$X \overset{d}{=} \psi(X, \Theta).$$

- The difference between ϕ and ψ is that the latter has positive probability of ignoring the specific value of X.

Markov Chain

Consider a Markov chain on stationarity $\{X_n\}_{n\in\mathbb{Z}}$ driven by the i.i.d. sequence $\{\Theta_n\}_{n\in\mathbb{Z}}$ satisfying

$$X_{n+1} \overset{d}{=} \psi\left(X_n, \Theta_n\right).$$

If we were able to find a time $-\tau < 0$ such that $X_{-\tau} \leq a\left(\Theta_{-\tau}\right)$, then

$$X_0 = \underbrace{\psi\Bigg(\cdots\psi}_{\tau-1\text{ times}}\Bigg(W_{-\tau}^{\frac{1}{\alpha\rho}}\left(1-U_{-\tau}\right)^{\frac{1}{\alpha}} S_{-\tau}, \Theta_{-\tau+1}\Bigg), \cdots, \Theta_{-1}\Bigg),$$

so we can compute $X_0 \sim \overline{S}\left(\alpha,\rho\right)$ from $\{\Theta_n\}_{n\in\{-\tau,\ldots,-1\}}$.

Dominating Process

Recall that if τ_n is the last time $\{X_{n-k} \leq a(\Theta_{n-k})\}$, then

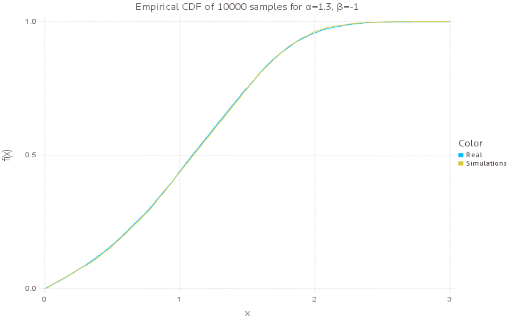
$$
\begin{aligned}
X_n &= \sum_{k=\tau_n+1}^{n-1} e^{\frac{1}{\alpha}\sum_{j=k+1}^{n-1}\log(\Lambda_j U_j)} \Lambda_k^{\frac{1}{\alpha}} (1-U_k)^{\frac{1}{\alpha}} S_k \\
&+ e^{\frac{1}{\alpha}\sum_{j=\tau_n+1}^{n-1}\log(\Lambda_j U_j)} W_\tau^{\frac{1}{\alpha\rho}} (1-U_\tau)^{\frac{1}{\alpha}} S_\tau \\
&\leq e^{R_n} \sum_{k=-\infty}^{n-1} e^{-(n-1-k)d} (1-U_k)^{\frac{1}{\alpha}} S_k \\
&\leq e^{R_n} \left(\frac{e^{(d-\delta)(\chi_n - n)}}{1-e^{\delta-d}} + \sum_{k=\chi_n}^{n-1} e^{-(n-1-k)d} (1-U_k)^{\frac{1}{\alpha}} S_k \right) =: D_n
\end{aligned}
$$

The Algorithm

1: Sample backwards in time $\{(D_n, \Theta_n)\}$ until $-\sigma$, the first time in which $\{D_n \leq a(\Theta_n)\}$
2: Put $X_{-\sigma+1} = \psi(a(\Theta_{-\sigma}), \Theta_{-\sigma})$
3: Compute recursively $X_n = \psi(X_{n-1}, \Theta_{n-1})$
4: **return** X_0 ▷ Here $X_0 \sim \overline{S}(\alpha, \rho)$

Sanity Check $(\alpha, \beta) = (1.3, -1)$

- Average sampling-time for each r.v.:0.011774 seconds
- Kolmogorov-Smirnov test *p*-value: 0.9213



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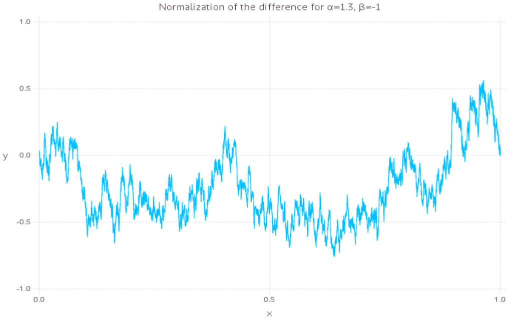
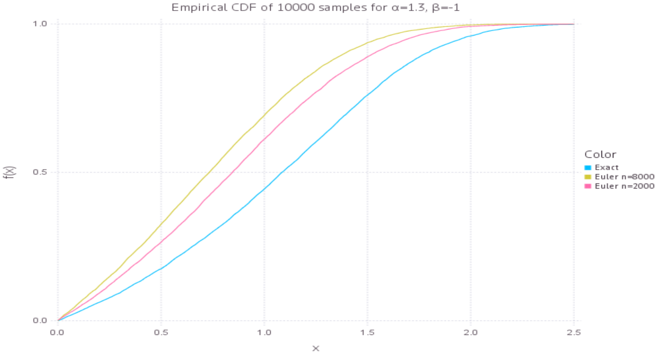


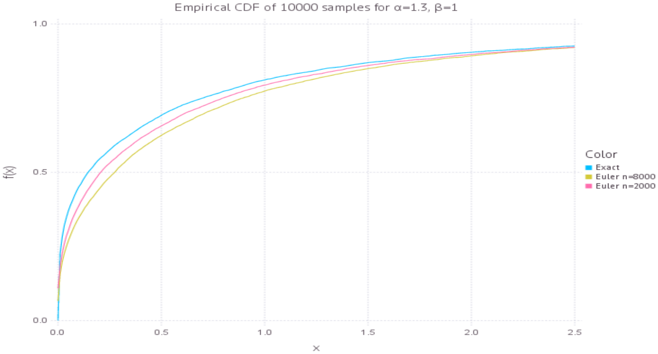
Discretisation $(\alpha, \beta) = (1.3, -1)$

- Kolmogorov distances for $N = 8,000$ and $2,000$ are 0.253 and 0.174 respectively



Discretisation $(\alpha, \beta) = (1.3, 1)$

- Kolmogorov distances for $N = 8,000$ and $2,000$ are 0.125 and 0.088 respectively



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[GCMUB18] Jorge I. González Cázares, Aleksandar Mijatović, and Gerónimo Uribe Bravo, *Exact simulation of the extrema of stable processes*.

[PUB12] Jim Pitman and Gerónimo Uribe Bravo, *The convex minorant of a Lévy process*, Ann. Probab. **40** (2012), no. 4, 1636–1674. MR 2978134