On the convergence of Island particle models

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Outline

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Notations

- $(\mathbb{X}_n, \mathcal{X}_n)_{n\geq 0}$ is a sequence of measurable sets.
- $\mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$ is the Banach space of all bounded and measurable functions on $(\mathbb{X}_n, \mathcal{X}_n)$.
- $(X_n)_{n\geq 0}$ is a non-homogenous Markov chain with initial distribution η_0, and Markov kernels $(M_n)_{n\geq 1}$.
- Feynman-Kac flow

$$\eta_n(f_n) \stackrel{\text{def}}{=} \gamma_n(f_n)/\gamma_n(1) \; ,$$

$$\gamma_n(f_n) \stackrel{\text{def}}{=} \mathbb{E}\left[f_n(X_n) \prod_{0\leq p<n} g_p(X_p)\right] \; .$$

Feynman-Kac flow

- Define by $\mathcal{P}(\mathbb{X}_n, \mathcal{X}_n)$ the set of probabilities on $(\mathbb{X}_n, \mathcal{X}_n)$.
- The sequence of probabilities $(\eta_n)_{n\geq 0}$ satisfies the following recursion:

$$\eta_{n+1} = \Psi_n(\eta_n)M_{n+1} \ ,$$

where $\Psi_n : \mathcal{P}(\mathbb{X}_n, \mathcal{X}_n) \to \mathcal{P}(\mathbb{X}_n, \mathcal{X}_n)$ is defined by:

$$\Psi_n(\eta_n)(A_n) \stackrel{\text{def}}{=} \frac{1}{\eta_n(g_n)} \int_{A_n} g_n(x_n)\ \eta_n(\mathrm{d}x_n) \ , \quad A_n \in \mathcal{X}_n \ .$$

Outline

Particle approximation

- Let N_1 be an integer. For any integer p we set $(\mathbb{X}_p, \boldsymbol{\mathcal{X}}_p) \stackrel{\text{def}}{=} (\mathbb{X}_p^{N_1}, \mathcal{X}_p^{\otimes N_1})$.
- Define the Markov kernel $\boldsymbol{M}_{n+1}$ from $(\mathbb{X}_n, \boldsymbol{\mathcal{X}}_n)$ to $(\mathbb{X}_{n+1}, \boldsymbol{\mathcal{X}}_{n+1})$ as the product measure

$$\boldsymbol{M}_{n+1}(\mathbf{x}_n, \mathbf{A}_{n+1}) \stackrel{\text{def}}{=} \prod_{1 \le i \le N_1} \Psi_n(m(\mathbf{x}_n, \cdot)) M_{n+1}(A_{n+1}^i) \ ,$$

where $m(\mathbf{x}_n, \cdot)$ stands for the empirical measure of $\mathbf{x}_n$ given for any $A_n \in \mathcal{X}_n$ by

$$m(\mathbf{x}_n, A_n) \stackrel{\text{def}}{=} \frac{1}{N_1} \sum_{i=1}^{N_1} \delta_{x_n^i}(A_n) \ .$$

- The particles are multinomially resampled with probabilities proportional to their potential $\{g_n(x_n^i)\}_{i=1}^{N_1}$; new particle positions are then sampled from the Markov kernel M_{n+1}.

Particle approximation

- Define a Markov chain $\{\boldsymbol{X}_n\}_{n\geq 0}$ where for each $n \in \mathbb{N}$,

$$\boldsymbol{X}_n = (\xi_n^1, \ldots, \xi_n^{N_1}) \in \mathbb{X}_n$$

with initial distribution $\boldsymbol{\eta}_0 \stackrel{\text{def}}{=} \eta_0^{\otimes N_1}$ and transition kernel $\boldsymbol{M}_{n+1}$.

- N_1-particle approximations

$$\eta_n^{N_1}(f_n) \stackrel{\text{def}}{=} m(\boldsymbol{X}_n, f_n)$$

$$\gamma_n^{N_1}(f_n) \stackrel{\text{def}}{=} \eta_n^{N_1}(f_n) \prod_{0\leq p<n} \eta_p^{N_1}(g_p) \, .$$

Unbiasedness of the particle approximation

Theorem (Del Moral, 199x)

For any $f_n \in \mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$, $\gamma_n^{N_1}(f_n)$ *is an unbiased estimator of* $\gamma_n(f_n)$*:*

$$\mathbb{E}\left[\gamma_n^{N_1}(f_n)\right] = \mathbb{E}\left[\eta_n^{N_1}(f_n) \prod_{0 \le p < n} \eta_p^{N_1}(g_p)\right] = \mathbb{E}\left[f_n(X_n) \prod_{0 \le p < n} g_p(X_p)\right] .$$

The island Feynman-Kac model

- For $\mathbf{x}_n = (x_n^1, \cdots, x_n^{N_1}) \in \mathbb{X}_n^{N_1}$ define the sample averaged potential

$$\boldsymbol{g}_n(\mathbf{x}_n) \stackrel{\text{def}}{=} m(\mathbf{x}_n, g_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} g_n(x_n^i) \; .$$

- Feynman-Kac model

$$\boldsymbol{\eta}_n(\boldsymbol{f}_n) = \boldsymbol{\gamma}_n(\boldsymbol{f}_n)/\boldsymbol{\gamma}_n(1)$$
$$\boldsymbol{\gamma}_n(\boldsymbol{f}_n) = \mathbb{E}\left[\boldsymbol{f}_n(\mathbf{X}_n) \prod_{0 \le p < n} \boldsymbol{g}_p(\mathbf{X}_p)\right] ,$$

The island Feynman-Kac model

Since $\boldsymbol{g}_n(\boldsymbol{X}_p) = \eta_n^{N_1}(g_p)$, the unbiasedness property implies that for any $\boldsymbol{f}_n$ of the form $\boldsymbol{f}_n(\mathbf{x}_n) = N_1^{-1}\sum_{i=1}^{N_1} f_n(x_n^i)$

$$\mathbb{E}\left[f_n(X_n)\prod_{0\leq p<n} g_p(X_p)\right] = \mathbb{E}\left[\boldsymbol{f}_n(\mathbf{X}_n)\prod_{0\leq p<n} \boldsymbol{g}_p(\mathbf{X}_p)\right] ,$$

or equivalently

$$\boldsymbol{\gamma}_n(\boldsymbol{f}_n) = \gamma_n(f_n) \quad \text{and} \quad \boldsymbol{\eta}_n(\boldsymbol{f}_n) = \eta_n(f_n) \; .$$

The island Feynman-Kac model

- From now on, a population of particles $\boldsymbol{X}_n$ is called an island.
- Idea: we may apply the interacting particle system approximation of the Feynman-Kac semigroups both within each island but also across island.
- To be more specific, we will now describe the so-called double bootstrap algorithm where the bootstrap algorithm is applied both within an island but also across the islands.
- Of course, many other options are available (more to come !)

Outline

Feynman-Kac at the island level

- Define by $\mathcal{P}(\mathbb{X}_n, \boldsymbol{\mathcal{X}}_n)$ the set of probabilities measures on $(\mathbb{X}_n, \boldsymbol{\mathcal{X}}_n)$.
- The sequence of measures $(\boldsymbol{\eta}_n)_{n\geq 0}$ satisfies the following recursion

$$\boldsymbol{\eta}_{n+1} = \boldsymbol{\Psi}_n(\boldsymbol{\eta}_n)\boldsymbol{M}_{n+1} \ ,$$

where $\boldsymbol{\Psi}_n : \mathcal{P}(\mathbb{X}_n, \boldsymbol{\mathcal{X}}_n) \to \mathcal{P}(\mathbb{X}_n, \boldsymbol{\mathcal{X}}_n)$ is defined by

$$\boldsymbol{\Psi}_n(\boldsymbol{\eta}_n)(\mathbf{A}_n) \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{\eta}_n(\boldsymbol{g}_n)} \int_{\mathbf{A}_n} \boldsymbol{g}_n(\mathbf{x}) \ \boldsymbol{\eta}_n(\mathrm{d}\mathbf{x}) \ , \quad \mathbf{A}_n \in \boldsymbol{\mathcal{X}}_n \ .$$

The double bootstrap algorithm

$$\left(\boldsymbol{\xi}_n^i\right) \xrightarrow{\text{selection}} \left(\widehat{\boldsymbol{\xi}}_n^i\right) \xrightarrow{\text{mutation}} \left(\boldsymbol{\xi}_{n+1}^i\right)$$

- Let N_2 be the number of interacting islands.
- During the selection stage, we select randomly N_2 islands $\left(\widehat{\boldsymbol{\xi}}_n^i\right)_{1\leq i\leq N_2}$ among the current islands $\left(\boldsymbol{\xi}_n^i\right)_{1\leq i\leq N_2} \in \mathbf{X}_n^{N_2}$ with probability proportional to the empirical mean of the individuals in each island

$$\boldsymbol{g}_n(\boldsymbol{\xi}_n^i) = N_1^{-1}\sum_{j=1}^{N_1} g_n(\xi_n^{i,j})\ , 1 \leq i \leq N_2\ .$$

- During the mutation transition, selected islands $(\widehat{\boldsymbol{\xi}}_n^i)_{i=1}^{N_2}$ evolve randomly to a new configuration $\boldsymbol{\xi}_{n+1}^i$ according to the Markov transition $\boldsymbol{M}_{n+1}$.

The double bootstrap

- Define the Markov kernel $\mathbf{L}_{n+1}^{N_2}$ from $(\mathbb{X}_n^{N_2}, \boldsymbol{\mathcal{X}}_n^{\otimes N_2})$ to $(\mathbb{X}_{n+1}^{N_2}, \boldsymbol{\mathcal{X}}_{n+1}^{\otimes N_2})$ for any $(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^{N_2}) \in \mathbb{X}_n^{N_2}$ and $(\mathbf{A}_{n+1}^1, \ldots, \mathbf{A}_{n+1}^{N_2}) \in \boldsymbol{\mathcal{X}}_n^{N_2}$ by

$$\mathbf{L}_{n+1}^{N_2}(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^{N_2}, \mathbf{A}_{n+1}^1 \times \cdots \times \mathbf{A}_{n+1}^{N_2}) \stackrel{\text{def}}{=} \prod_{1 \le i \le N_2} \boldsymbol{\Psi}_n(\boldsymbol{m}(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^{N_2}, \cdot))\boldsymbol{M}_{n+1}(\mathbf{A}_{n+1}^i) \, ,$$

where $\boldsymbol{m}(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^{N_2}, \cdot)$ stands for the empirical measure of the islands $(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^{N_2})$ given for any $\mathbf{A}_n \in \boldsymbol{\mathcal{X}}_n$ by

$$\boldsymbol{m}(\mathbf{x}_n^1, \ldots, \mathbf{x}_n^{N_2}, \mathbf{A}_n) \stackrel{\text{def}}{=} \frac{1}{N_2} \sum_{i=1}^{N_2} \delta_{\mathbf{x}_n^i}(\mathbf{A}_n) \, .$$

The double bootstrap algorithm

for p from 0 to $n-1$ **do**

  Sample $\boldsymbol{I}_p = (I_p^i)_{i=1}^{N_2}$ multinomially with proba. prop. to $\left(\frac{1}{N_1}\sum_{j=1}^{N_1} g_p(\xi_p^{i,j})\right)_{i=1}^{N_2}$.

  for i from 1 to N_2 **do**

    Sample $\boldsymbol{J}_p^i = (J_p^{i,j})_{j=1}^{N_1}$ multinomially with proba. prop. to $\left(g_p(\xi_p^{I_p^i,j})\right)_{j=1}^{N_1}$.

    For $1 \leq j \leq N_1$, sample independently $\xi_{p+1}^{i,j}$ according to $M_{p+1}(\xi_p^{I_p^i, J_p^j}, \cdot)$.

  end for

end for

Bootstrap approximation: bias and variance

Theorem

For any time horizon $n \geq 0$ *and any bounded function* $f_n \in \mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$, *we have*

$$\lim_{N_1 \to \infty} N_1 \mathbb{E}\left[\eta_n^{N_1}(f_n) - \eta_n(f_n)\right] = B_n(f_n) ,$$

$$\lim_{N_1 \to \infty} N_1 \mathbb{V}\mathrm{ar}\left(\eta_n^{N_1}(f_n)\right) = V_n(f_n) ,$$

where $B_n(f_n)$ *and* $V_n(f_n)$ *can be computed explicitly.*

Double bootstrap approximation: bias and variance

Theorem

For any time horizon $n \geq 0$ and any $f_n \in \mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$, we have

$$\lim_{N_1\to\infty} \lim_{N_2\to\infty} N_1 N_2 \mathbb{E}\left[\boldsymbol{\eta}_n^{N_2}(m(\cdot, f_n)) - \boldsymbol{\eta}_n(m(\cdot, f_n))\right] = B_n(f_n) + \widetilde{B}_n(f_n) \; ,$$

$$\lim_{N_1\to\infty} \lim_{N_2\to\infty} N_1 N_2 \mathbb{V}\mathrm{ar}\left(\boldsymbol{\eta}_n^{N_2}(m(\cdot, f_n))\right) = V_n(f_n) + \widetilde{V}_n(f_n) \; ,$$

where $B_n(f_n)$, $\widetilde{B}_n(f_n)$, $V_n(f_n)$, $\widetilde{V}_n(f_n)$ can be computed explicitly.

- The rate of the interacting island (N_2 islands each with N_1 individuals) is the same as the one of the single island model with $N_1 N_2$ particles.
- Even though the constant terms may be worst in the interacting island model, it allows to use parallel implementations.

Independent islands

Theorem

For any time horizon $n \geq 0$ and any $f_n \in \mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$, we have

$$\lim_{N_1 \to \infty} N_1 \left\{ \mathbb{E}\left[\widetilde{\boldsymbol{\eta}}_n^{N_2}(m(\cdot, f_n)) \right] - \eta_n(f_n) \right\} = B_n(f_n) \;,$$

$$\lim_{N_1 \to \infty} N_1 N_2 \mathbb{V}\mathrm{ar}\left(\widetilde{\boldsymbol{\eta}}_n^{N_2}(m(\cdot, f_n)) \right) = V_n(f_n) \;,$$

where $B_n(f_n)$ and $V_n(f_n)$ are the same than for the single island model.

Although the variance of the particle approximation is inversely proportional to $N_1 N_2$, the bias is independent of N_2 and is inversely proportional to N_1.

Example

1. Linear Gaussian Model

- $X_{p+1} = \phi X_p + \sigma_u U_p$,
- $Y_p = X_p + \sigma_v V_p$,

Computing the predictive distribution of the state X_n given the observations $Y_{0:n-1} = y_{0:n-1}$ up to time $n-1$ can be cast into the framework of Feynman-Kac model by setting for all $p \geq 0$

$$M_{p+1}(x_p, \mathrm{d}x_{p+1}) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left[-(x_{p+1} - \phi x_p)^2/(2\sigma_u^2)\right] \mathrm{d}x_{p+1} ,$$

$$g_p(x_p) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left[-(y_p - x_p)^2/(2\sigma_v^2)\right] .$$

How to choose between interacting and independent islands?

	Independent islands	Interacting islands
Squared bias	$\dfrac{B_n(f_n)^2}{N_1^2}$	$\dfrac{\left(B_n(f_n)+\widetilde{B}_n(f_n)\right)^2}{N_1^2 N_2^2}$
Variance	$\dfrac{V_n(f_n)}{N_1 N_2}$	$\dfrac{V_n(f_n)+\widetilde{V}_n(f_n)}{N_1 N_2}$
Sum	$\dfrac{V_n(f_n)}{N_1 N_2}+\dfrac{B_n(f_n)^2}{N_1^2}$	$\dfrac{V_n(f_n)+\widetilde{V}_n(f_n)}{N_1 N_2}$

$$\frac{V_n(f_n)}{N_1 N_2}+\frac{B_n(f_n)^2}{N_1^2} < \frac{V_n(f_n)+\widetilde{V}_n(f_n)}{N_1 N_2} \quad \Leftrightarrow \quad N_1 > \frac{B_n(f_n)^2}{\widetilde{V}_n(f_n)} N_2 \ .$$

Numerical application: Linear Gaussian Model

- The model is defined by

$$X_{p+1} = \phi X_p + \sigma_u U_p \ , \quad Y_p = X_p + \sigma_v V_p \ .$$

- $n + 1 = 11$ observations were generated with $\phi = 0.9$, $\sigma_u = 0.6$ and $\sigma_v = 1$.
- We have $\mathbb{E}\left[X_n | Y_{0:n-1} = y_{0:n-1}\right] = \eta_n(\mathrm{Id})$. M
- We compare interacting to independent islands through

$$100 \times \frac{\mathbb{E}\left[\left(\boldsymbol{\eta}_n^{N_2}(\mathrm{Id}) - \eta_n(\mathrm{Id})\right)^2\right] - \mathbb{E}\left[\left(\widetilde{\boldsymbol{\eta}}_n^{N_2}(\mathrm{Id}) - \eta_n(\mathrm{Id})\right)^2\right]}{\mathbb{E}\left[\left(\widetilde{\boldsymbol{\eta}}_n^{N_2}(\mathrm{Id}) - \eta_n(\mathrm{Id})\right)^2\right]} \ .$$

Results for the LGSS model

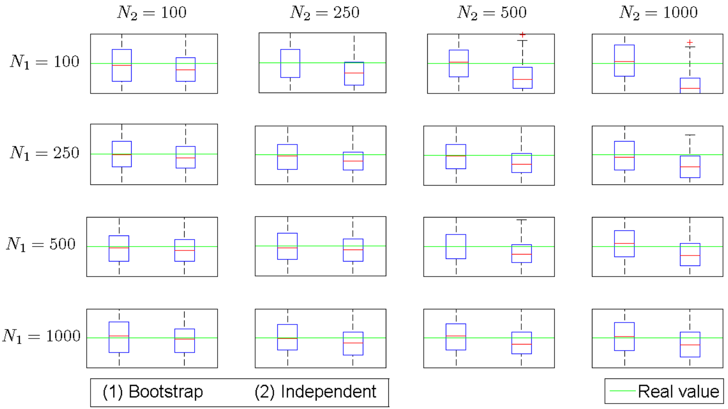


Figure: Interacting versus independent island renormalized estimators.

Outline

Effective Sample Size Interaction

- Define

$$\Theta_{n,\alpha} = \left\{ \mathbf{x}_n = (x_n^1, w_n^1, \ldots, x_n^{N_1}, w_n^{N_1}) \in \mathbb{X}_n \middle| \frac{\left(\sum_{i=1}^{N_1} w_n^i g_n(x_n^i)\right)^2}{\sum_{i=1}^{N_1} \left(w_n^i g_n(x_n^i)\right)^2} \geq \alpha N_1 \right\} .$$

- Define $m(\mathbf{x}_n, \cdot)$ stands for the empirical measure of $\mathbf{x}_n$ given for any $A_n \in \mathcal{X}_n$ by

$$m(\mathbf{x}_n, A_n) \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{N_1} w_n^i} \sum_{i=1}^{N_1} w_n^i \delta_{x_n^i}(A_n) ,$$

Effective Sample Size Interaction

Consider the Markov kernel $\boldsymbol{M}_{n+1}$

$$\boldsymbol{M}_{n+1}(\mathbf{x}_n, \mathbf{A}_{n+1}) = \begin{cases} \prod_{i=1}^{N_1} \delta_{w_n^i g_n(x_n^i)}(B_{n+1}^i) M_{n+1}(x_n^i, A_{n+1}^i) & \mathbf{x}_n \in \Theta_{n,\alpha} \\ \prod_{i=1}^{N_1} \delta_1(B_{n+1}^i) \Psi_n(m(\mathbf{x}_n, \cdot)) M_{n+1}(A_{n+1}^i) & \mathbf{x}_n \notin \Theta_{n,\alpha} \end{cases}$$

Define a Markov chain $\{\boldsymbol{X}_n\}_{n \geq 0}$ where for each $n \in \mathbb{N}$,

$$\boldsymbol{X}_n = \left[(\xi_n^1, \omega_n^1), \ldots, (\xi_n^{N_1}, \omega_n^{N_1})\right] \in \mathbb{X}_n \ ,$$

ESS: particle approximation

N_1-particle approximations of the measures η_n and γ_n

$$\eta_n^{N_1}(f_n) \stackrel{\text{def}}{=} m(\boldsymbol{X}_n, f_n) = \frac{1}{\sum_{i=1}^{N_1} \omega_n^i} \sum_{i=1}^{N_1} \omega_n^i f_n\left(\xi_n^i\right) ,$$

$$\gamma_n^{N_1}(f_n) \stackrel{\text{def}}{=} \eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p) .$$

Theorem

For any $f_n \in \mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$, $\gamma_n^{N_1}(f_n)$ is an unbiased estimator of $\gamma_n(f_n)$:

$$\mathbb{E}\left[\gamma_n^{N_1}(f_n)\right] = \mathbb{E}\left[\eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p)\right] = \mathbb{E}\left[f_n(X_n) \prod_{0 \leq p < n} g_p(X_p)\right] .$$

ESS: Feynman-Kac approximation

- For $\mathbf{x}_n = (x_n^1, w_n^1, \cdots, x_n^{N_1}, w_n^{N_1}) \in \mathbb{X}_n$ we set

$$\boldsymbol{g}_n(\mathbf{x}_n) \stackrel{\text{def}}{=} m(\mathbf{x}_n, g_n) = \frac{1}{\sum_{i=1}^{N_1} w_n^i} \sum_{i=1}^{N_1} w_n^i g_n\left(x_n^i\right) .$$

- The associated Feynman-Kac model $\{(\boldsymbol{\eta}_n, \boldsymbol{\gamma}_n)\}_{n\geq 0}$ is

$$\boldsymbol{\eta}_n(\boldsymbol{f}_n) = \boldsymbol{\gamma}_n(\boldsymbol{f}_n)/\boldsymbol{\gamma}_n(1)$$
$$\boldsymbol{\gamma}_n(\boldsymbol{f}_n) = \mathbb{E}\left[\boldsymbol{f}_n(\mathbf{X}_n) \prod_{0\leq p<n} \boldsymbol{g}_p(\mathbf{X}_p)\right] ,$$

ESS: Feynman-Kac approximation

Since $\boldsymbol{g}_n(\boldsymbol{X}_n) = \eta_n^{N_1}(g_n)$, for any $\boldsymbol{f}_n$ of the form $\boldsymbol{f}_n(\mathbf{x}_n) = \left(\sum_{i=1}^{N_1} w_n^i\right)^{-1} \sum_{i=1}^{N_1} w_n^i f_n\left(x_n^i\right)$ where $f_n \in \mathcal{B}_b(\mathbb{X}_n, \mathcal{X}_n)$,

$$\mathbb{E}\left[f_n(X_n) \prod_{0 \le p < n} g_p(X_p)\right] = \mathbb{E}\left[\boldsymbol{f}_n(\mathbf{X}_n) \prod_{0 \le p < n} \boldsymbol{g}_p(\mathbf{X}_p)\right] ,$$

Therefore

$$\boldsymbol{\gamma}_n(\boldsymbol{f}_n) = \gamma_n(f_n)$$
$$\boldsymbol{\eta}_n(\boldsymbol{f}_n) = \eta_n(f_n) \ .$$

for p **from** 0 **to** $n-1$ **do**

    Selection step and weight actualization between islands:

    Set $N_2^{\text{eff}} = \left(\sum_{i=1}^{N_2} \Omega_p^i \boldsymbol{g}_p(\boldsymbol{\xi}_p^i, \boldsymbol{\omega}_p^i)\right)^2 / \sum_{i=1}^{N_2} \left(\Omega_p^i \boldsymbol{g}_p(\boldsymbol{\xi}_p^i, \boldsymbol{\omega}_p^i)\right)^2$.

    if $N_2^{\text{eff}} \geq \alpha_{\text{Islands}} N_2$ **then**

        For $1 \leq i \leq N_2$, **set** $\Omega_{p+1}^i = \Omega_p^i \boldsymbol{g}_p(\boldsymbol{\xi}_p^i, \boldsymbol{\omega}_p^i)$.

        Set $\boldsymbol{I}_p = (I_p^i)_{i=1}^{N_2} = (1, 2, \dots, N_2)$.

    else

        Set $\boldsymbol{\Omega}_{p+1} = \left(\Omega_{p+1}^i\right)_{i=1}^{N_2} = (1, \dots, 1)$.

        Sample $\boldsymbol{I}_p = (I_p^i)_{i=1}^{N_2}$ **multinomially with proba. prop. to** $\left(\Omega_p^i \boldsymbol{g}_p(\boldsymbol{\xi}_p^i, \boldsymbol{\omega}_p^i)\right)_{i=1}^{N_2}$.

    end if

    Island mutation step:

    for i **from** 1 **to** N_2 **do**

        Particle selection and weight actualization within each island:

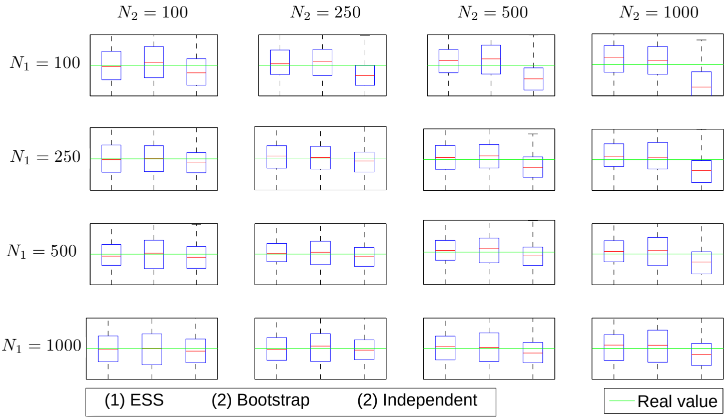
        same business as usual

    end for

end for

Approximate $\eta_n(f_n)$ **by** $\dfrac{1}{\sum_{i=1}^{N_2} \Omega_n^i} \sum_{i=1}^{N_2} \Omega_n^i \dfrac{1}{\sum_{j=1}^{N_1} \omega_n^{i,j}} \sum_{j=1}^{N_1} \omega_n^{i,j} f_n\left(\xi_n^{i,j}\right)$.

Results for the ESS model



Number of interactions

Table: Number of interactions between islands for the ESS within ESS estimator as a percentage of the one the ESS within bootstrap estimator in the LGM.

N_1 \ N_2	100	250	500	1000
100	4.32	4.76	4.92	4.98
250	0.88	0.60	0.34	0.32
500	0.04	0.02	0	0
1000	0	0	0	0