Filtering for discretely observed jump diffusions

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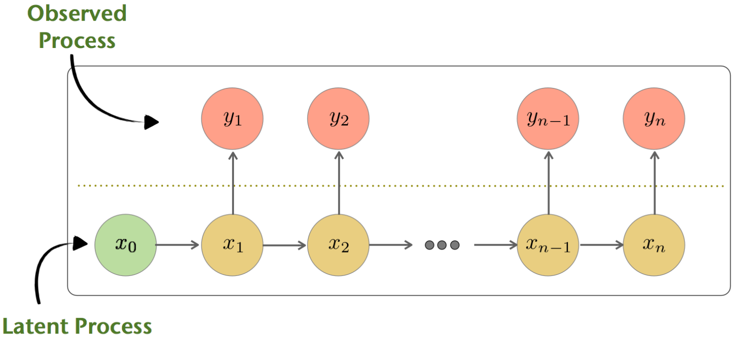
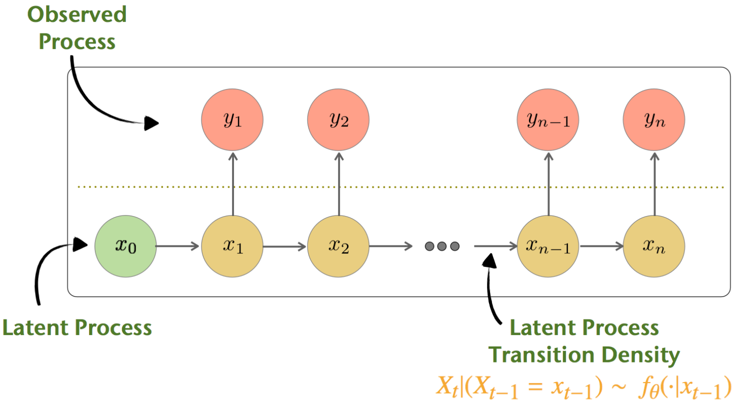
Problem Outline

Framework I

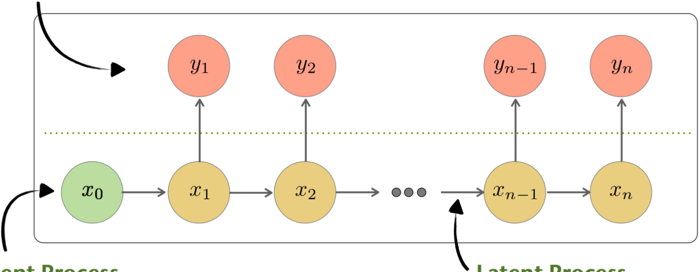


Framework II



Framework III

Observed Process



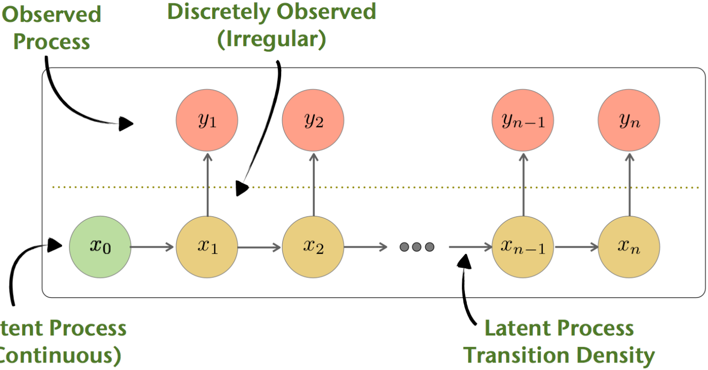
Latent Process

Latent Process Transition Density

$$X_t|(X_{t-1} = x_{t-1}) \sim f_\theta(\cdot|x_{t-1})$$

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

Framework IV



Observed Process

Discretely Observed (Irregular)

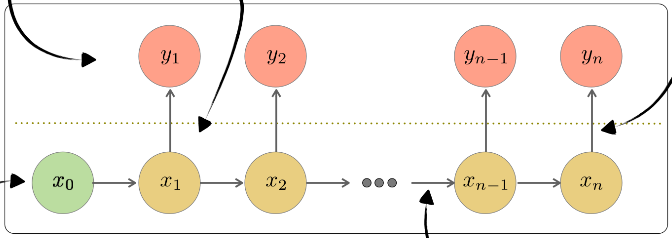
Latent Process (Continuous)

Latent Process Transition Density

$$X_t|(X_{t-1} = x_{t-1}) \sim f_\theta(\cdot|x_{t-1})$$

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

Framework V



Observed Process

Discretely Observed (Irregular)

Observed Process Density

$Y_t|(X_t = x_t) \sim g_\theta(\cdot|x_t)$

Latent Process (Continuous)

Latent Process Transition Density

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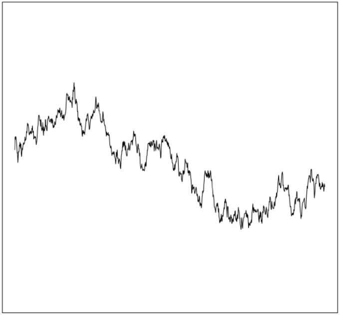
What is a Diffusion? I

Diffusion

$$\xrightarrow{} \mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$



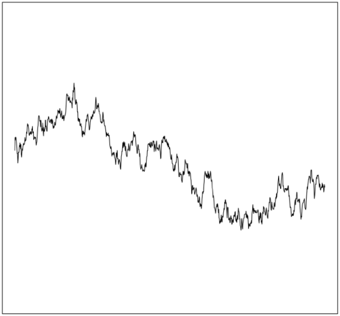
What is a Diffusion? II

Diffusion

Instantaneous Mean

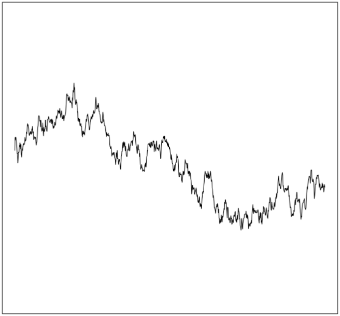
Time

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

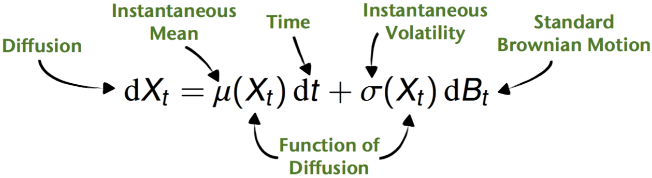


What is a Diffusion? III

$$\overset{\text{Diffusion}}{\mathrm{d}X_t} = \overset{\text{Instantaneous Mean}}{\mu(X_t)}\,\overset{\text{Time}}{\mathrm{d}t} + \overset{\text{Instantaneous Volatility}}{\sigma(X_t)}\,\overset{\text{Standard Brownian Motion}}{\mathrm{d}B_t}$$



What is a Diffusion? IV



Diffusion

Instantaneous Mean

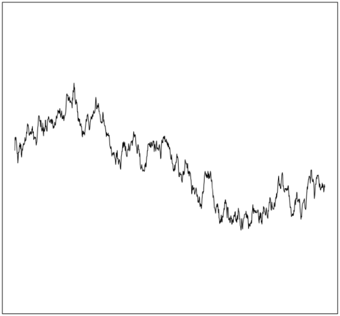
Time

Instantaneous Volatility

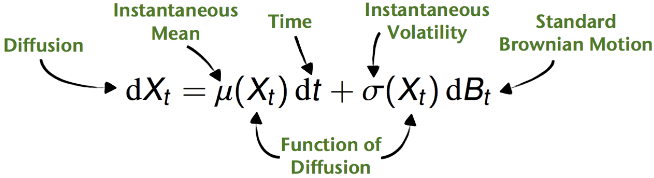
Standard Brownian Motion

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

Function of Diffusion



What is a Diffusion? V



$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

Diffusion

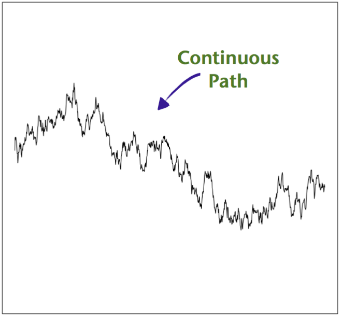
Instantaneous Mean

Time

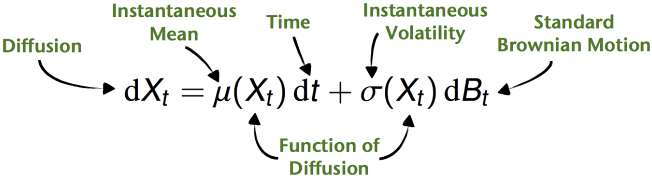
Instantaneous Volatility

Standard Brownian Motion

Function of Diffusion



What is a Diffusion? VI



Diffusion

Instantaneous Mean

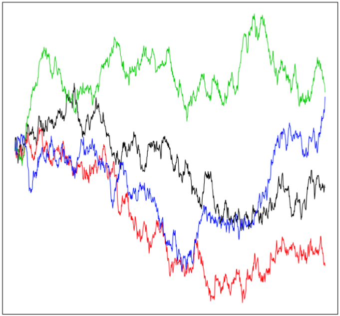
Time

Instantaneous Volatility

Standard Brownian Motion

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

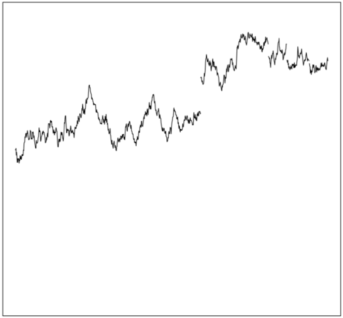
Function of Diffusion



What is a Diffusion? VII

Jump Diffusion

$$\overset{\text{Jump Diffusion}}{\downarrow}\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

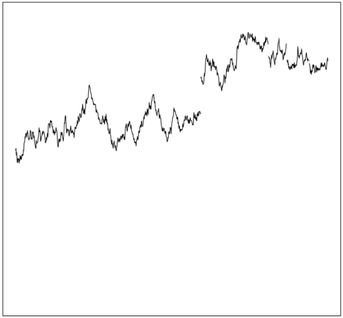


What is a Diffusion? VIII

Jump Diffusion

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

Poisson Jump Process



What is a Diffusion? IX

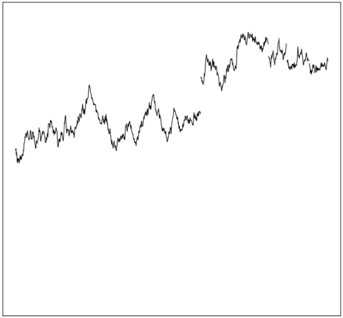
Jump Diffusion

Jump Intensity

Jump Size Density

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

Poisson Jump Process



What is a Diffusion? X

Jump Diffusion

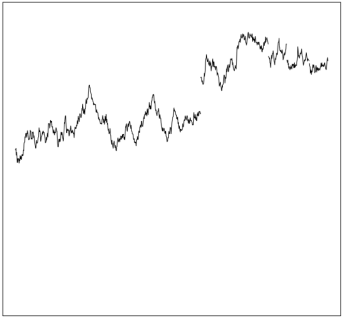
Jump Intensity

Jump Size Density

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

Poisson Jump Process

Function of Diffusion



What is a Diffusion? XI

Jump Diffusion

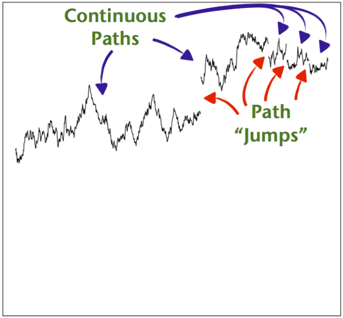
Jump Intensity

Jump Size Density

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

Poisson Jump Process

Function of Diffusion



What is a Diffusion? XII

Jump Diffusion

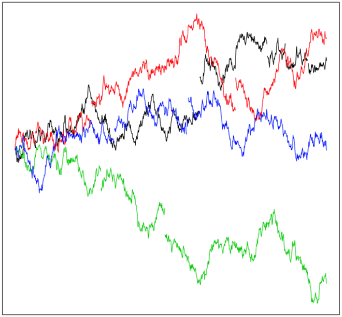
Jump Intensity

Jump Size Density

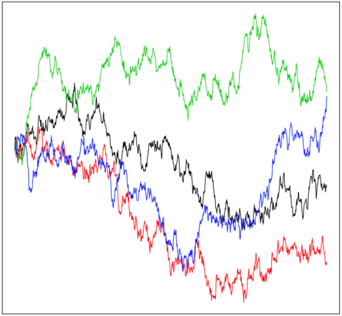
$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right),\nu\left(X_{t-}\right)\right]$$

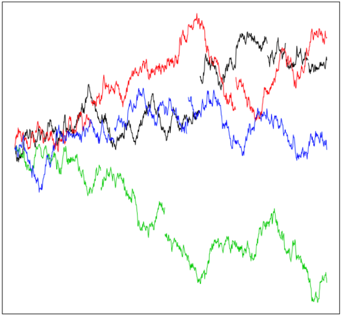
Poisson Jump Process

Function of Diffusion

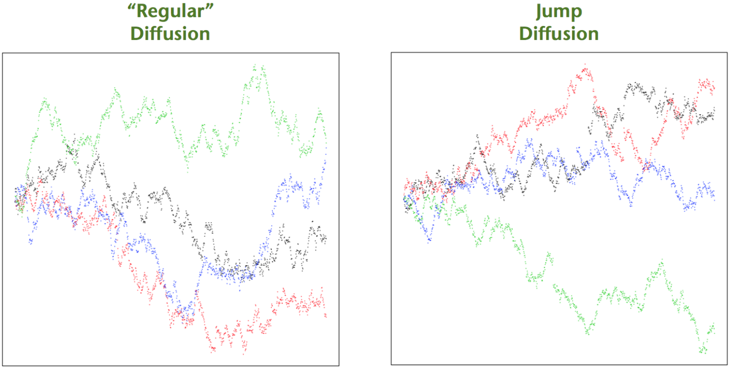


What is a Diffusion? XIII


“Regular” Diffusion


Jump Diffusion

What is a Diffusion? XIV



The Problem

- **Key Problem**

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    - Not possible to simulate entire diffusion sample paths. . .

The Problem

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  - Not possible to simulate entire diffusion sample paths. . .
  - . . . Finite representation.

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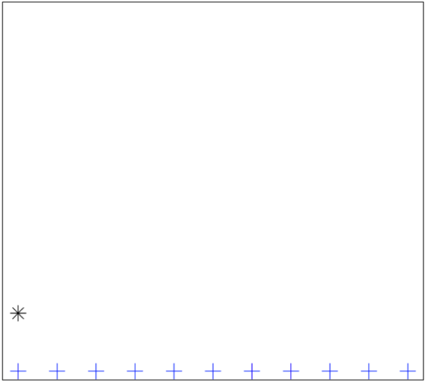
The Problem

- **Key Problem**
  - Not possible to simulate entire diffusion sample paths. . .
  - . . . Finite representation.
  - . . . Transition density inaccessible.
- **What can we simulate?**
  - . . .

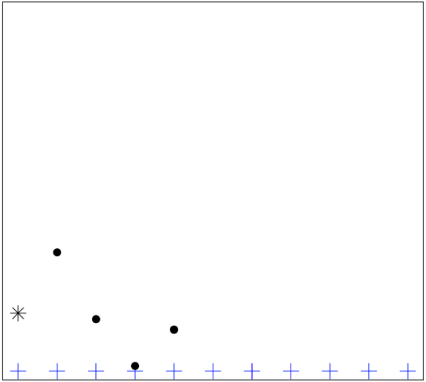
What can we simulate? I

$$\mathrm{d}X_t = \mathrm{d}B_t$$



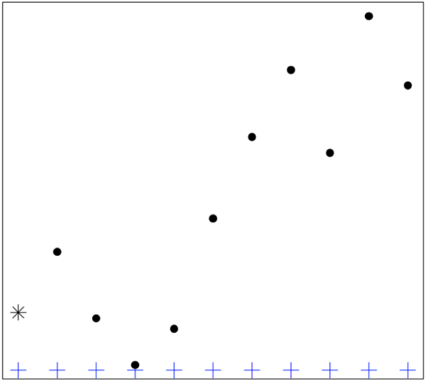
What can we simulate? II

$$\mathrm{d}X_t = \mathrm{d}B_t$$



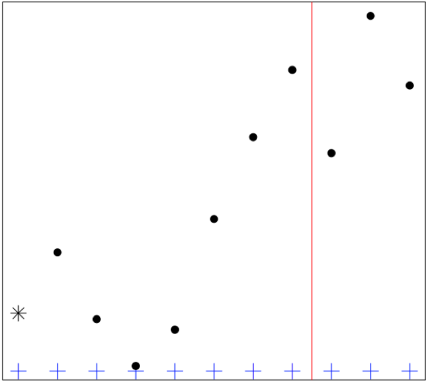
What can we simulate? III

$$\mathrm{d}X_t = \mathrm{d}B_t$$



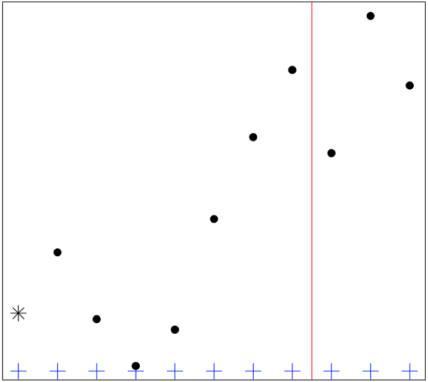
What can we simulate? IV

$$\mathrm{d}X_t = \mathrm{d}B_t$$



What can we simulate? V

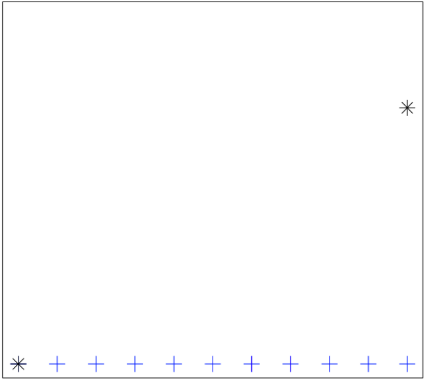
$$\mathrm{d}X_t = \mathrm{d}B_t$$



“sufficiency”

What can we simulate? VI

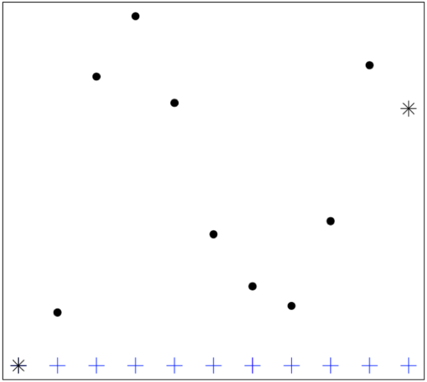
$$dX_t = dB_t$$



“sufficiency”

What can we simulate? VII

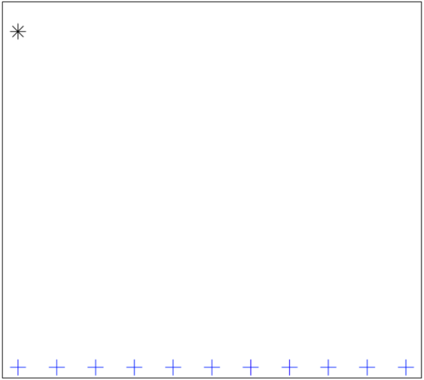
$$\mathrm{d}X_t = \mathrm{d}B_t$$



"sufficiency"

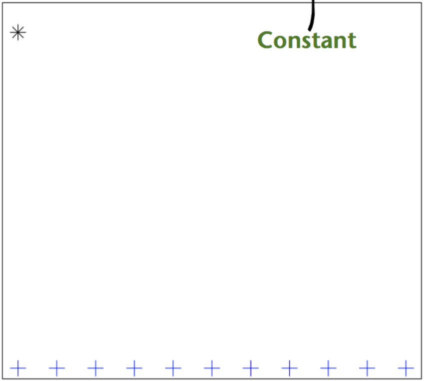
What can we simulate? VIII

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda, \nu\left(X_{t-}\right)\right]$$



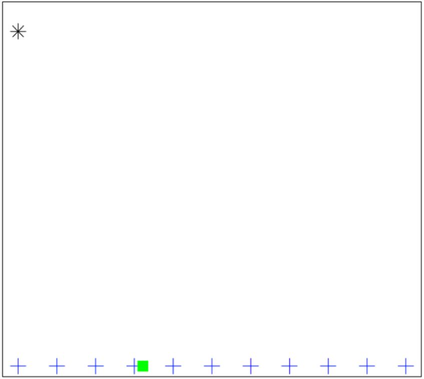
What can we simulate? IX

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\underset{\uparrow\ \text{Constant}}{\lambda}, \nu\left(X_{t-}\right)\right]$$



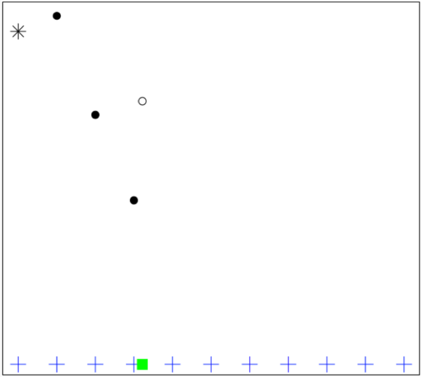
What can we simulate? X

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda, \nu\left(X_{t-}\right)\right]$$



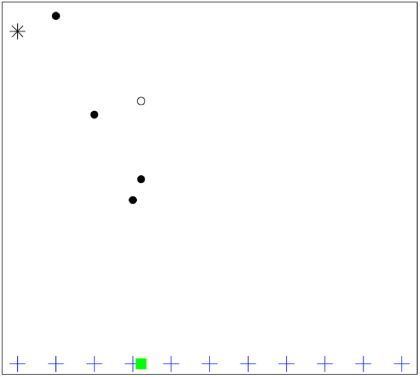
What can we simulate? XI

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda, \nu\left(X_{t-}\right)\right]$$



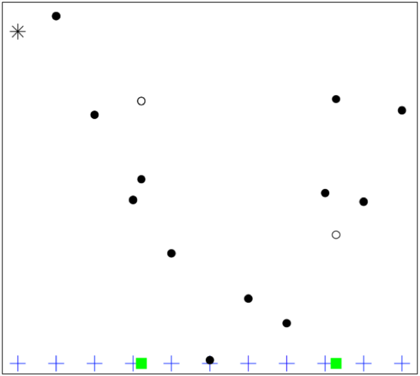
What can we simulate? XII

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda, \nu\left(X_{t-}\right)\right]$$



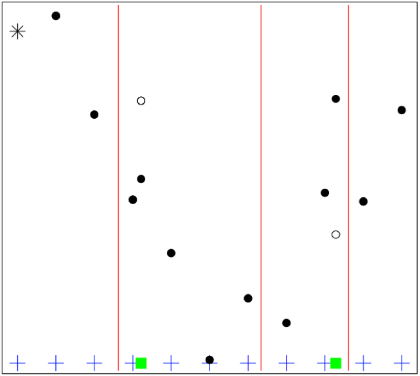
What can we simulate? XIII

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda, \nu\left(X_{t-}\right)\right]$$



What can we simulate? XIV

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda, \nu\left(X_{t-}\right)\right]$$

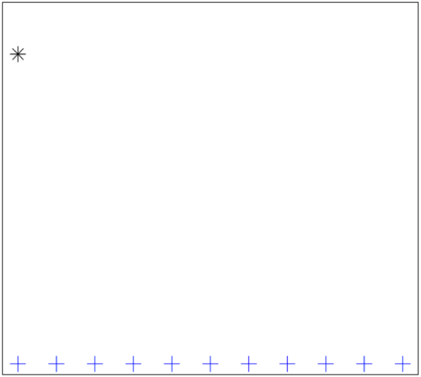


What can we simulate? XV

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

$$\uparrow$$

$$\sup_{t\in[0,T]} \lambda(X_t) \leq \Lambda < \infty$$



What can we simulate? XVI

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

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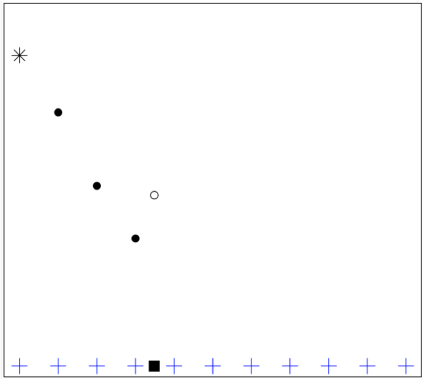
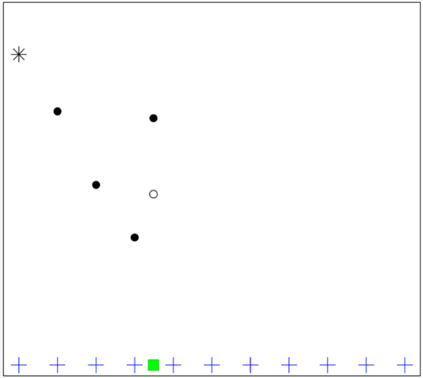


What can we simulate? XVII

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

$$\uparrow$$

$$\sup_{t\in[0,T]} \lambda(X_t) \leq \Lambda < \infty$$

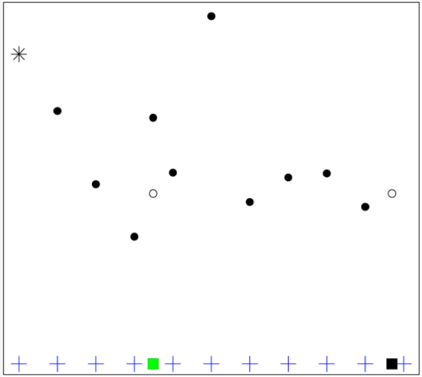


What can we simulate? XVIII

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

$$\uparrow$$

$$\sup_{t\in[0,T]} \lambda(X_t) \leq \Lambda < \infty$$

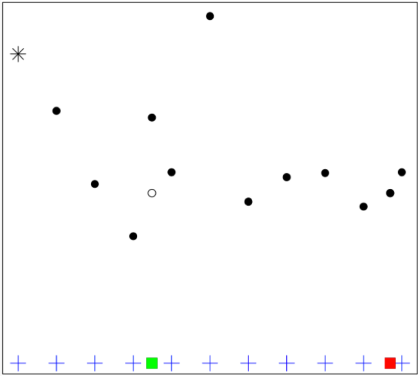


What can we simulate? XIX

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

$$\uparrow$$

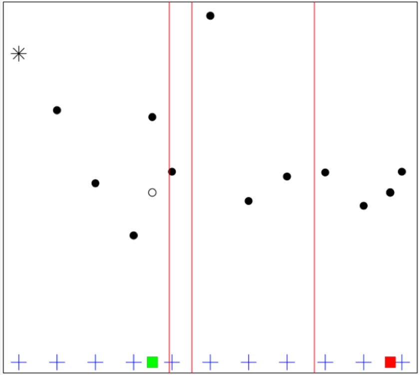
$$\sup_{t\in[0,T]} \lambda(X_t) \leq \Lambda < \infty$$



What can we simulate? XX

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}J_{t-}\left[\lambda\left(X_{t-}\right), \nu\left(X_{t-}\right)\right]$$

$$\mathrm{sup}_{t\in[0,T]}\, \lambda(X_t) \leq \Lambda < \infty$$



Discretisation

- **What about everything else?**

Discretisation

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    - “Discretize” diffusion dynamics.

Discretisation

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  - “Discretize” diffusion dynamics.
    - Euler discretization, for instance (PJS09, Jav05).

$$X_{t+\Delta t} = \begin{cases} \mu(X_t)\,\Delta t + \sigma(X_t)\,\mathcal{N}(0,\Delta t) & \text{w.p. } e^{-\lambda(X_{t-})\Delta t} \\ \mu(X_t)\,\Delta t + \sigma(X_t)\,\mathcal{N}(0,\Delta t) + \nu(X_{t-}) & \text{w.p. } 1 - e^{-\lambda(X_{t-})\Delta t} \end{cases}$$

Discretisation

- **What about everything else?**
  - “Discretize” diffusion dynamics.
    - Euler discretization, for instance (PJS09, Jav05).

$$X_{t+\Delta t} = \begin{cases} \mu\left(X_t\right)\Delta t + \sigma\left(X_t\right)\mathcal{N}(0,\Delta t) & \text{w.p. } e^{-\lambda(X_{t-})\Delta t} \\ \mu\left(X_t\right)\Delta t + \sigma\left(X_t\right)\mathcal{N}(0,\Delta t) + \nu\left(X_{t-}\right) & \text{w.p. } 1 - e^{-\lambda(X_{t-})\Delta t} \end{cases}$$

    - Miltstein, Shoji & Ozaki,. . .

Discretisation

- **What about everything else?**
  - "Discretize" diffusion dynamics.
    - Euler discretization, for instance (PJS09, Jav05).

$$X_{t+\Delta t} = \begin{cases} \mu\left(X_t\right)\Delta t + \sigma\left(X_t\right)\mathcal{N}(0,\Delta t) & \text{w.p. } e^{-\lambda(X_{t-})\Delta t} \\ \mu\left(X_t\right)\Delta t + \sigma\left(X_t\right)\mathcal{N}(0,\Delta t) + \nu\left(X_{t-}\right) & \text{w.p. } 1 - e^{-\lambda(X_{t-})\Delta t} \end{cases}$$

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  - Drawbacks?

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  - Drawbacks?
    - Implicit error.
    - Computationally expensive.
    - Moving beyond ‘bootstrap’?
    - Performance evaluated using (finely discretised) simulated data.

Talk Outline

- Problem Outline & Motivation.

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- Area Estimator

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- Exact Algorithm (BPRF06, BPR08, PJR12a)

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- Other (Related) Work. (PJR12a, PJR12b)
- Questions (& Hopefully) Answers.

Area Estimator

Expectations of Positive RVs

Goal…

- Evaluate $\mathbb{E}(P)$ for some r.v. $P \in \mathbb{R}_+$

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Suppose for now. . .

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- $u \sim U[0, 1]$

Expectations of Positive RVs

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Suppose for now. . .

- $P \in [0, 1]$
- $u \sim U[0, 1]$

Consider a biased coin C_p . . .

- Heads (1) if $u \leq P$
- Tails (0) if $u > P$

Expectations of Positive RVs

Why is this interesting. . . ?

$$\begin{aligned}
\mathbb{P}\Big(C_p = 1\Big) &= \mathbb{E}\Big(\mathbb{1}\Big(u \leq P\Big)\Big) && \text{-}\ \mathbb{P}(X) = \mathbb{E}(\mathbb{1}_X) \\
&= \mathbb{E}\Big(\mathbb{E}\Big(\mathbb{1}\Big(u \leq P\Big)\Big|P\Big)\Big) && \text{- Tower Property} \\
&= \mathbb{E}\Big(\mathbb{P}\Big(C_p = 1\Big|P\Big)\Big) && \text{-}\ \mathbb{P}(X) = \mathbb{E}(\mathbb{1}_X) \\
&= \mathbb{E}\Big(P\Big)
\end{aligned}$$

To estimate $\mathbb{E}\Big(P\Big)$ unbiasedly we can simply throw a C_p coin

Retrospective Area Estimator

- **Let's consider an example...** Suppose we want to evaluate,

$$\mathbb{E}\big(P\big) := \mathbb{E}\left[\exp\left\{-\int_0^T f(t)\,\mathrm{d}t\right\}\right]$$

Retrospective Area Estimator

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$$\mathbb{P}\big(\mathcal{N} = 0\big) = \exp\bigg\{-\int_0^T f(t)\,\mathrm{d}t\bigg\}$$

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- Consider a Poisson process with instantaneous rate $f(t)$ on $[0, T]$,

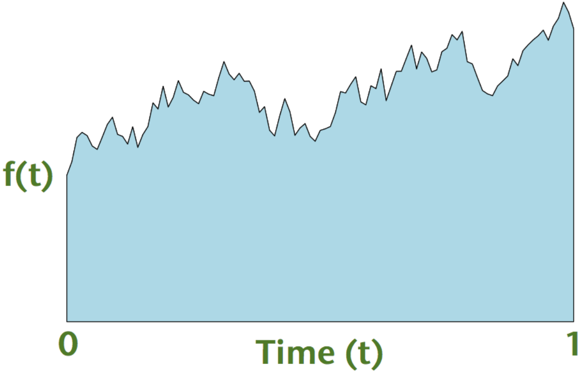
$$\mathbb{P}\Big(\mathcal{N} = 0\Big) = \exp\left\{-\int_0^T f(t)\,\mathrm{d}t\right\}$$

- $\mathbb{P}\Big(\mathcal{N} = 0\Big) \equiv P = \mathbb{E}\Big(P\Big)$

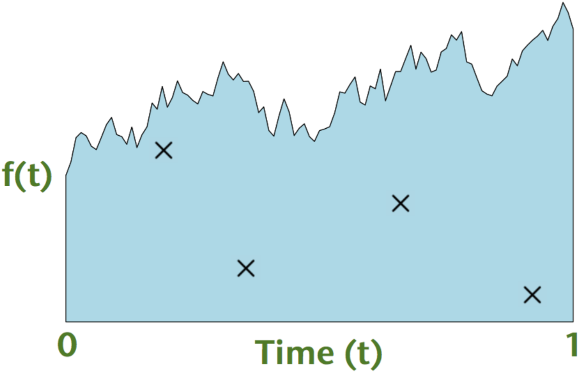
Retrospective Area Estimator I

The algorithm. . .

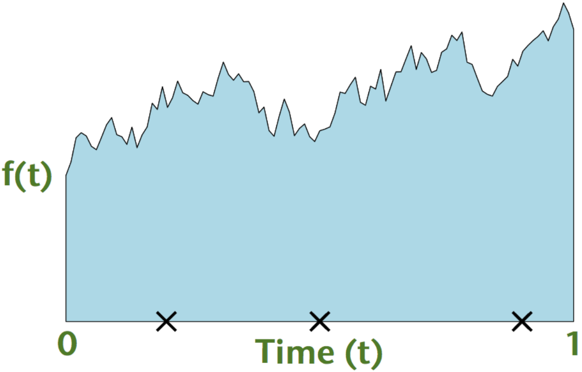
Retrospective Area Estimator II



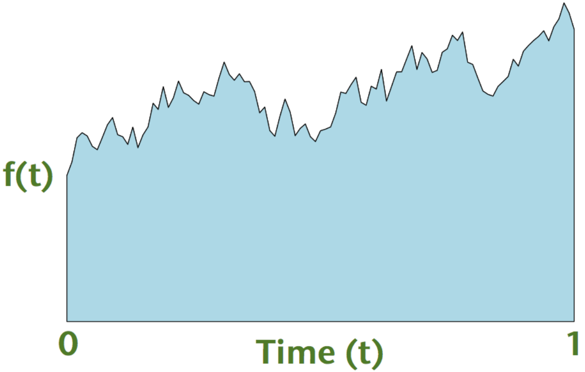
Retrospective Area Estimator III



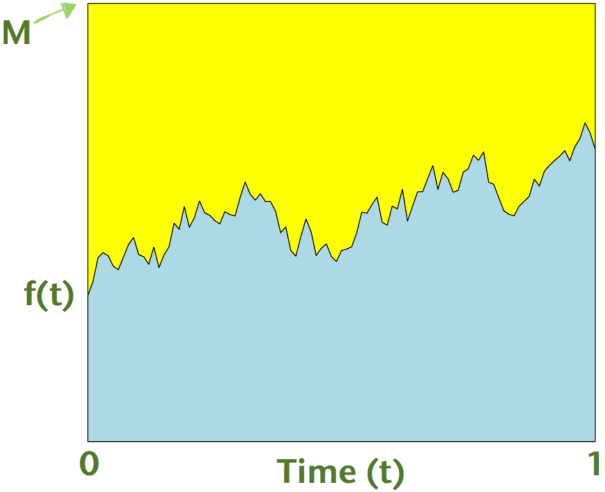
Retrospective Area Estimator IV



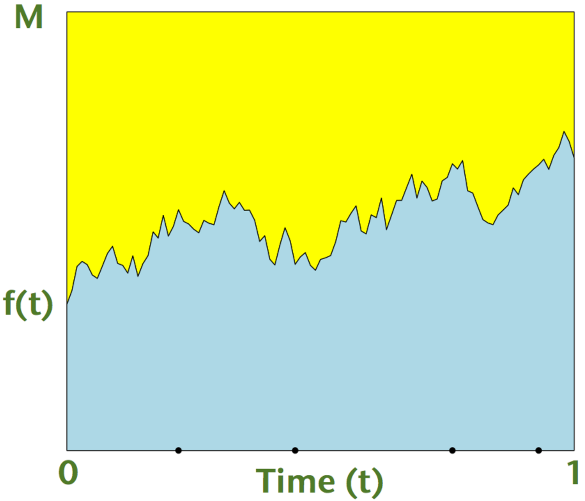
Retrospective Area Estimator V



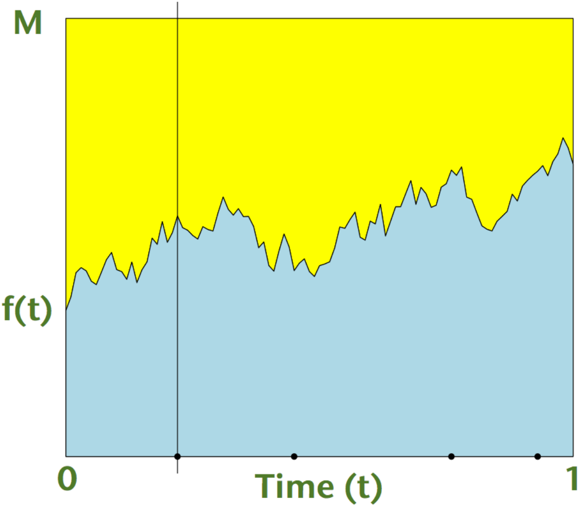
Retrospective Area Estimator VI



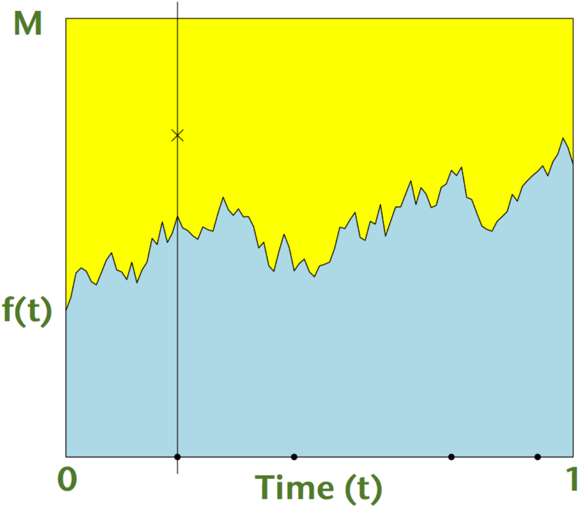
Retrospective Area Estimator VII



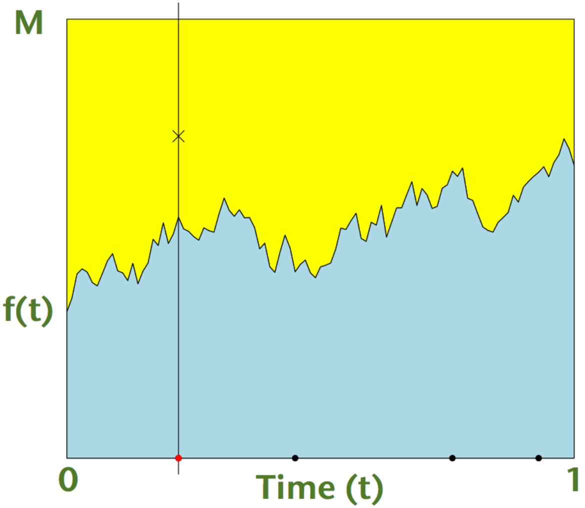
Retrospective Area Estimator VIII



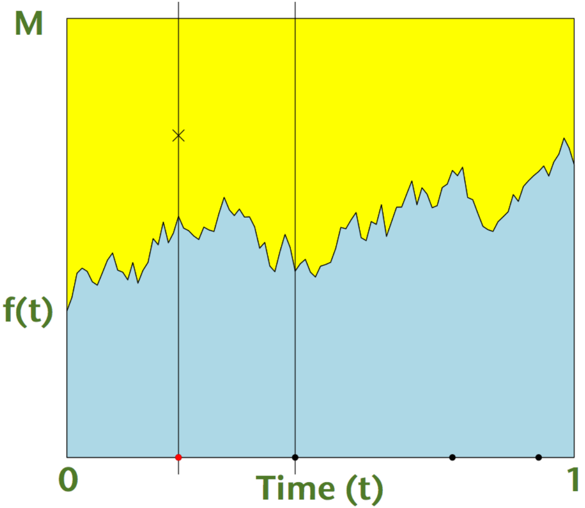
Retrospective Area Estimator IX



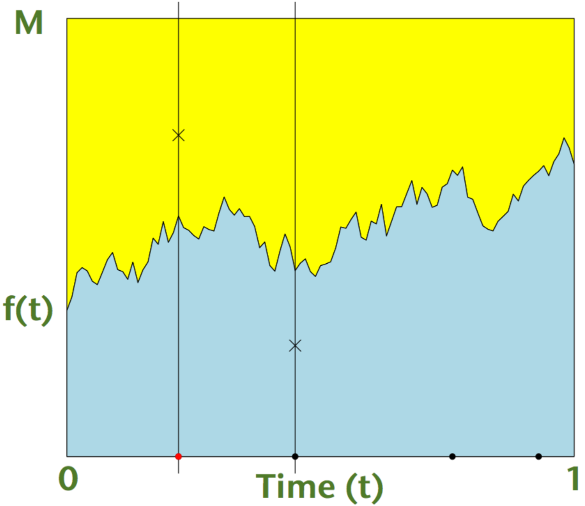
Retrospective Area Estimator X



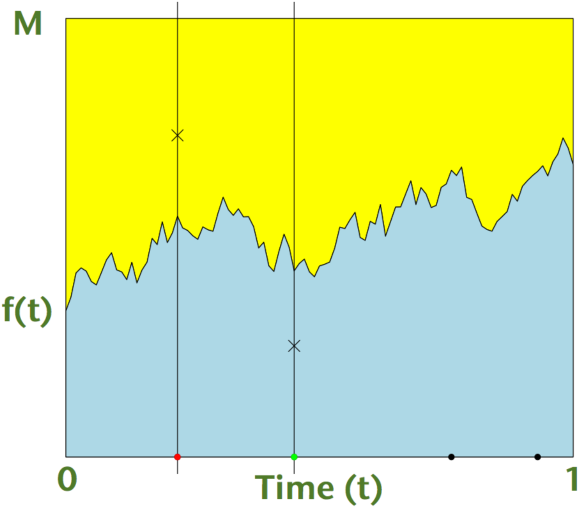
Retrospective Area Estimator XI



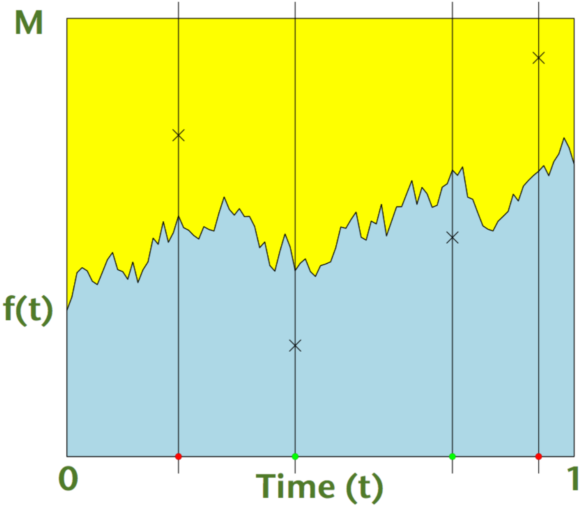
Retrospective Area Estimator XII



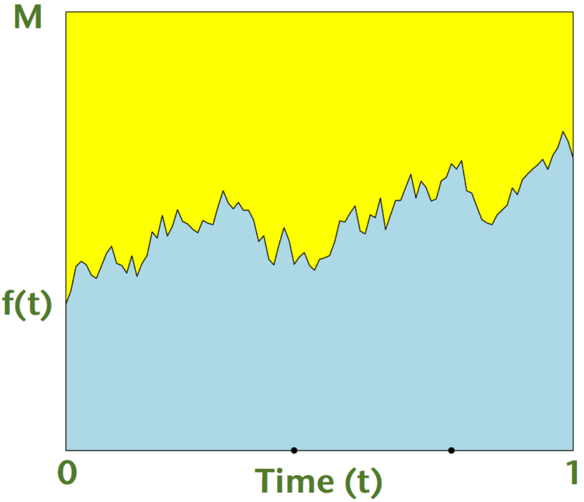
Retrospective Area Estimator XIII



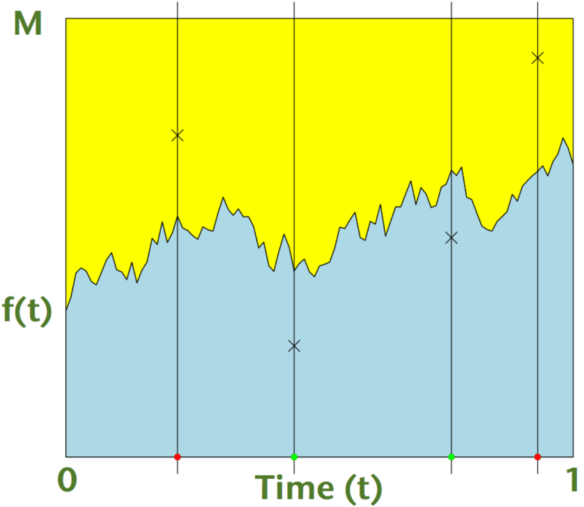
Retrospective Area Estimator XIV



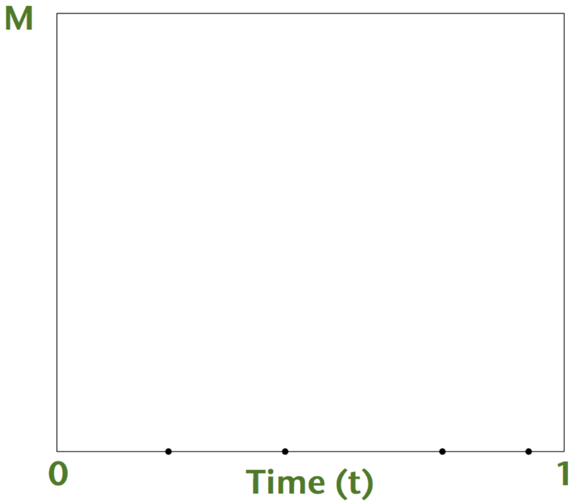
Retrospective Area Estimator XV



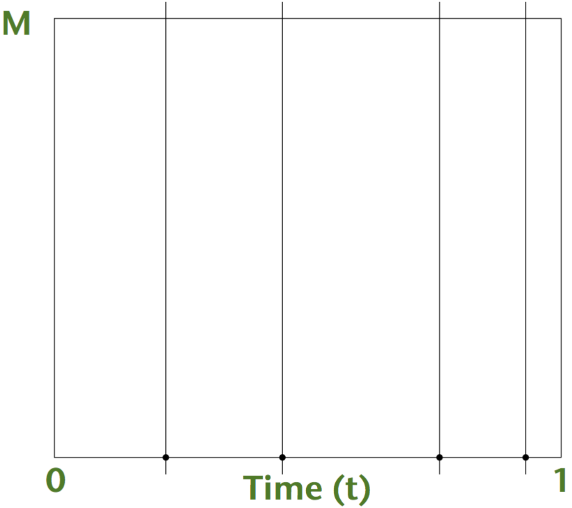
Retrospective Area Estimator XVI



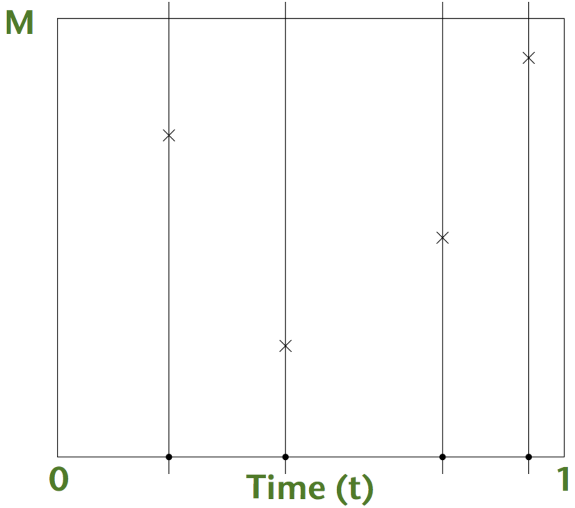
Retrospective Area Estimator XVII



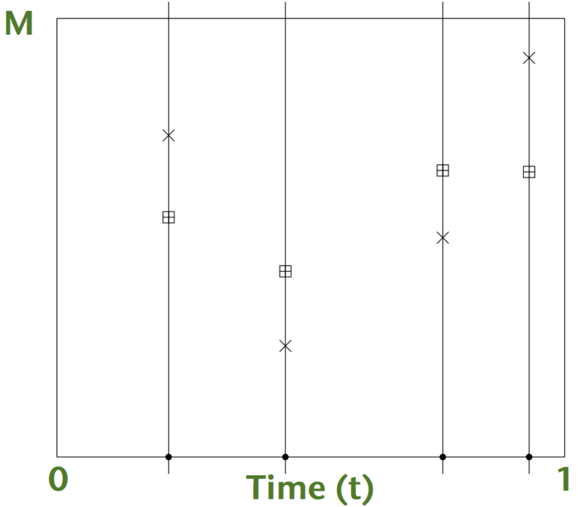
Retrospective Area Estimator XVIII



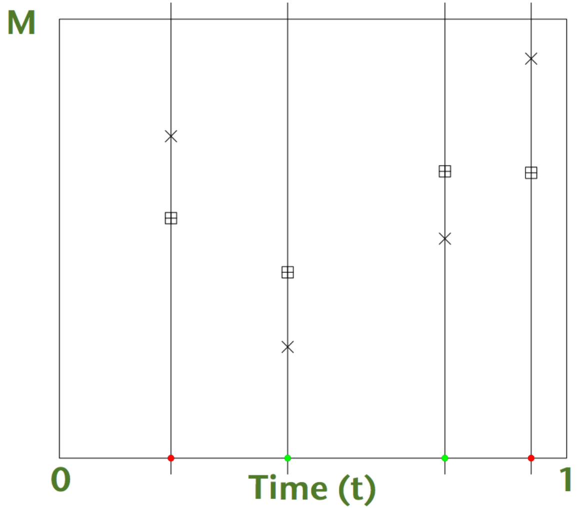
Retrospective Area Estimator XIX



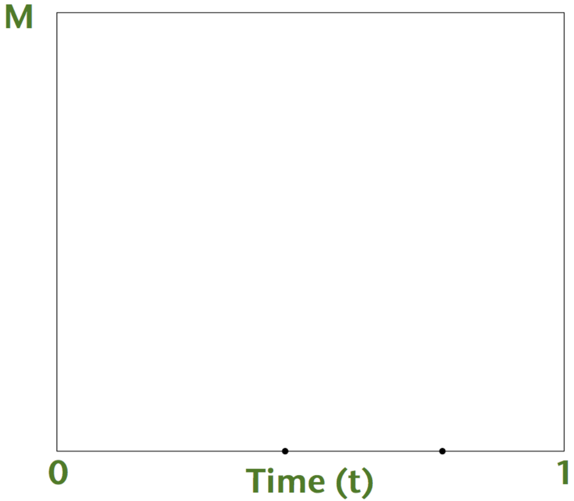
Retrospective Area Estimator XX



Retrospective Area Estimator XXI



Retrospective Area Estimator XXII



Transition Density

Transition Denisty

- **First consider no jumps**

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  - Transform Diffusion - *Lamperti Transform*

$$\mathrm{d}X_t = \alpha(X_t)\,\mathrm{d}t + \mathrm{d}B_t$$

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  - Propose sample paths (which can be simulated) - *Brownian motion*.

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    - Absolutely continuous.

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    - Absolutely continuous.
  - Accept or reject.

Transition Density I

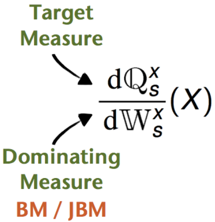
$$\frac{\mathrm{d}\mathbb{Q}_s^x}{\mathrm{d}\mathbb{W}_s^x}(X)$$

Transition Density II

Target Measure

$$\frac{\mathrm{d}\mathbb{Q}_s^x}{\mathrm{d}\mathbb{W}_s^x}(X)$$

Transition Density III



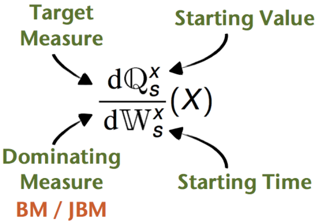
Target Measure

$$\frac{\mathrm{d}\mathbb{Q}_s^x}{\mathrm{d}\mathbb{W}_s^x}(X)$$

Dominating Measure

BM / JBM

Transition Density IV



Target Measure

Starting Value

$$\frac{\mathrm{d}\mathbb{Q}_s^x}{\mathrm{d}\mathbb{W}_s^x}(X)$$

Dominating Measure

BM / JBM

Starting Time

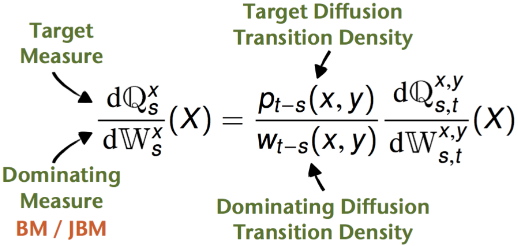
Transition Density V

Target
Measure

$$\frac{\mathrm{d}\mathbb{Q}^x_s}{\mathrm{d}\mathbb{W}^x_s}(X) = \frac{p_{t-s}(x,y)}{w_{t-s}(x,y)}\,\frac{\mathrm{d}\mathbb{Q}^{x,y}_{s,t}}{\mathrm{d}\mathbb{W}^{x,y}_{s,t}}(X)$$

Dominating
Measure
BM / JBM

Transition Density VI



Target Measure

Target Diffusion Transition Density

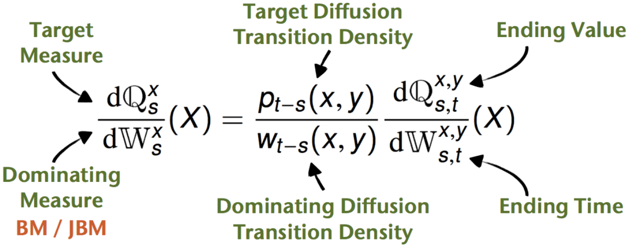
$$\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X) = \frac{p_{t-s}(x,y)}{w_{t-s}(x,y)} \frac{\mathrm{d}\mathbb{Q}^{x,y}_{s,t}}{\mathrm{d}\mathbb{W}^{x,y}_{s,t}}(X)$$

Dominating Measure

BM / JBM

Dominating Diffusion Transition Density

Transition Density VII



Target Measure

Target Diffusion Transition Density

Ending Value

$$\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X) = \frac{p_{t-s}(x,y)}{w_{t-s}(x,y)} \frac{\mathrm{d}\mathbb{Q}^{x,y}_{s,t}}{\mathrm{d}\mathbb{W}^{x,y}_{s,t}}(X)$$

Dominating Measure

BM / JBM

Dominating Diffusion Transition Density

Ending Time

Transition Density VIII

$$\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X)\right] = \mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{p_{t-s}(x,y)}{w_{t-s}(x,y)}\frac{\mathrm{d}\mathbb{Q}^{x,y}_{s,t}}{\mathrm{d}\mathbb{W}^{x,y}_{s,t}}(X)\right]$$

Expectation wrt
Conditioned
Dominating Measure

Transition Density IX

$$\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X)\right] = \mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{p_{t-s}(x,y)}{w_{t-s}(x,y)}\,\frac{\overset{1}{\overset{\swarrow}{\mathrm{d}\mathbb{Q}^{x,y}_{s,t}}}}{\mathrm{d}\mathbb{W}^{x,y}_{s,t}}(X)\right]$$

Expectation wrt
Conditioned
Dominating Measure

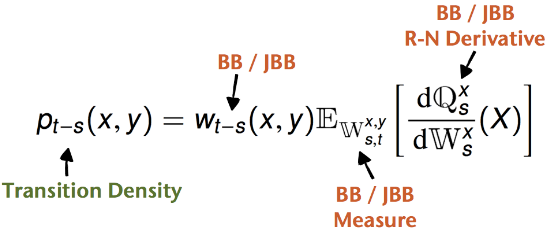
Transition Density X

$$p_{t-s}(x,y) = w_{t-s}(x,y)\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{\mathrm{d}\mathbb{Q}^x_s}{\mathrm{d}\mathbb{W}^x_s}(X)\right]$$

↑

Transition Density

Transition Density XI



$$p_{t-s}(x,y) = w_{t-s}(x,y)\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X)\right]$$

BB / JBB
R-N Derivative

BB / JBB

Transition Density

BB / JBB
Measure

Transition Density XII

BB / JBB
R-N Derivative

BB / JBB

$$p_{t-s}(x,y) = w_{t-s}(x,y)\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X)\right]$$

Transition Density

BB / JBB
Measure

Transition Density XIII

Brownian Bridge
- No Jumps
Dachuna-Castelle

$$p_{t-s}(x,y) = \mathcal{N}_{t-s}\,(y-x)\exp\{A(X_t)-A(X_s)-\ell(t-s)\}\;\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]$$

$\uparrow$

Transition Density

Transition Density XIV

Brownian Bridge
- No Jumps
Dachuna-Castelle

Normal pdf

Modified end point

$$p_{t-s}(x,y) = \mathcal{N}_{t-s}(y-x)\exp\{A(X_t) - A(X_s) - \ell(t-s)\}\ \mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]$$

Transition Density

Transition Density XV

Brownian Bridge
- No Jumps
Dachuna-Castelle

$$\underset{\substack{\uparrow\\ \text{Transition Density}}}{p_{t-s}(x,y)} = \overset{\substack{\text{Normal pdf}\\ \downarrow}}{\mathcal{N}_{t-s}}(y-x)\exp\{A(X_t) \overset{\substack{\text{Modified end point}\\ \downarrow}}{-} A(X_s) - \ell(t-s)\}\ \mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]$$

where, $\phi(X_u) := \dfrac{\alpha^2(X_u) + \alpha'(X_u)}{2} - \ell$

Transition Density XVI

Brownian Bridge
- No Jumps
Dachuna-Castelle

$$p_{t-s}(x,y) = \underbrace{\mathcal{N}_{t-s}\,(y-x)\exp\{A(X_t)-A(X_s)-\ell(t-s)\}}_{h_{t-s}(x,y)}\ \mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]$$

↑

Transition Density

where, $\phi(X_u) := \dfrac{\alpha^2(X_u)+\alpha'(X_u)}{2} - \ell$

Transition Density XVII

Brownian Bridge
- No Jumps
Dachuna-Castelle

$$p_{t-s}(x,y) = \underbrace{\mathcal{N}_{t-s}\,(y-x)\exp\{A(X_t)-A(X_s)-\ell(t-s)\}}_{h_{t-s}(x,y)}\;\underbrace{\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]}_{\text{weight}}$$

↑

Transition Density

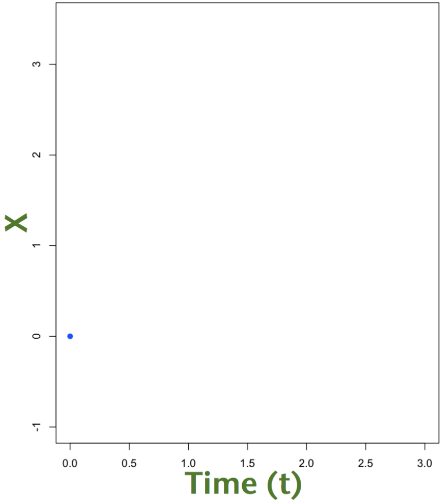
where, $\phi(X_u) := \dfrac{\alpha^2(X_u)+\alpha'(X_u)}{2} - \ell$

Exact Algorithm

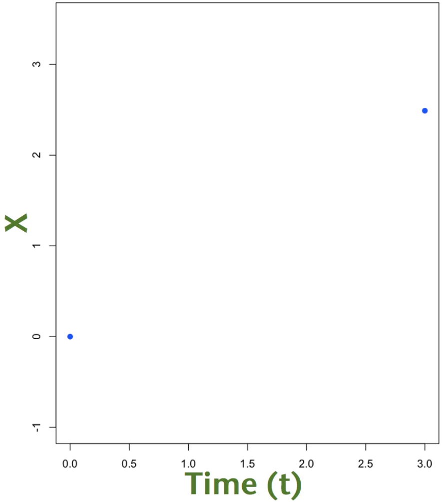
Exact Algorithm

1. Simulate end point - $y \sim h$
2. Propose sample bridge
3. Accept or Reject proposed sample bridge (and GOTO 1)

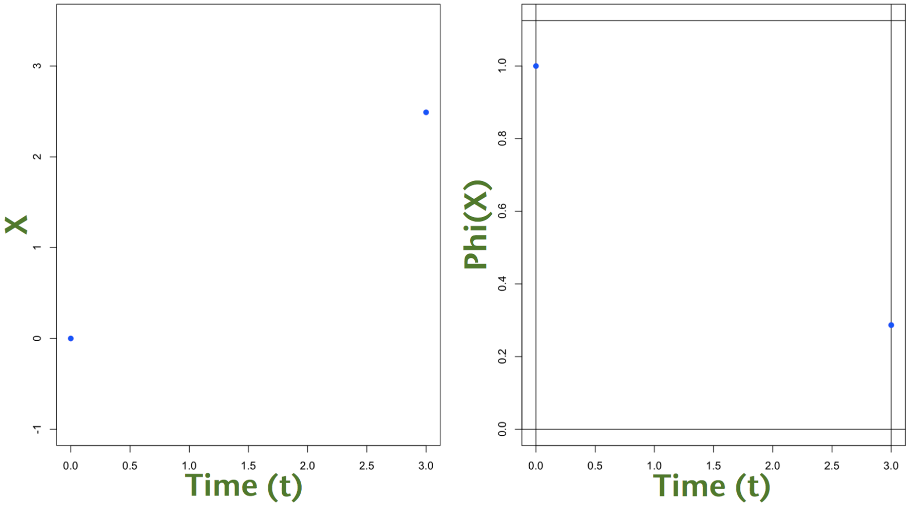
Exact Algorithm I



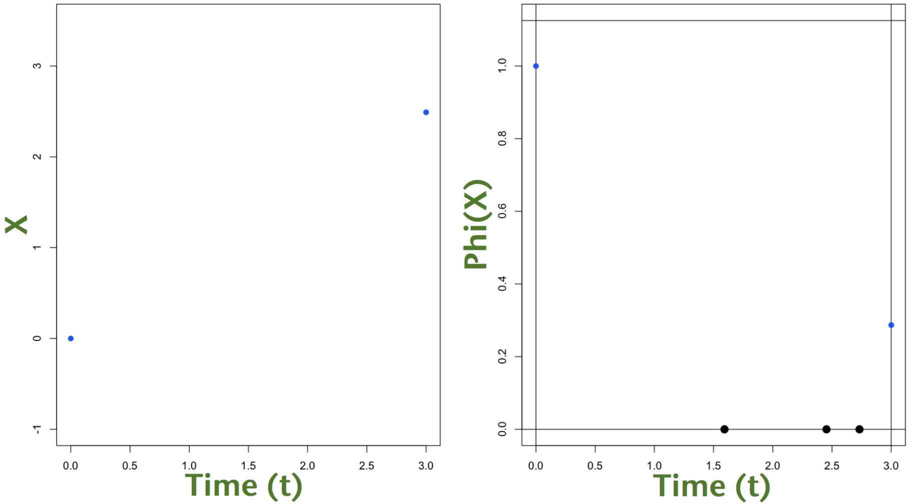
Exact Algorithm II



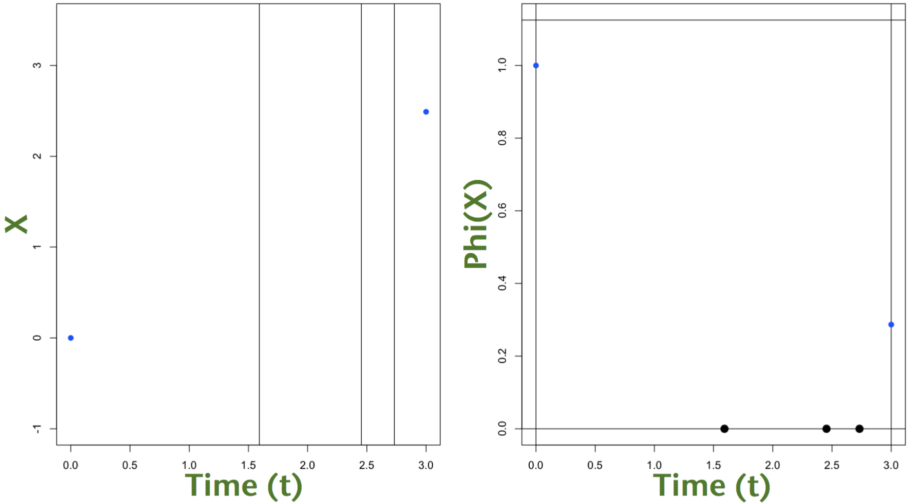
Exact Algorithm III



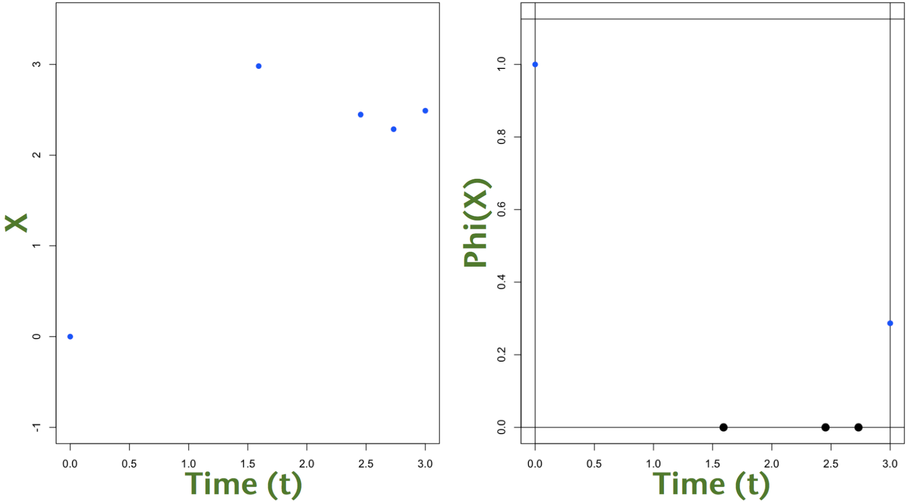
Exact Algorithm IV



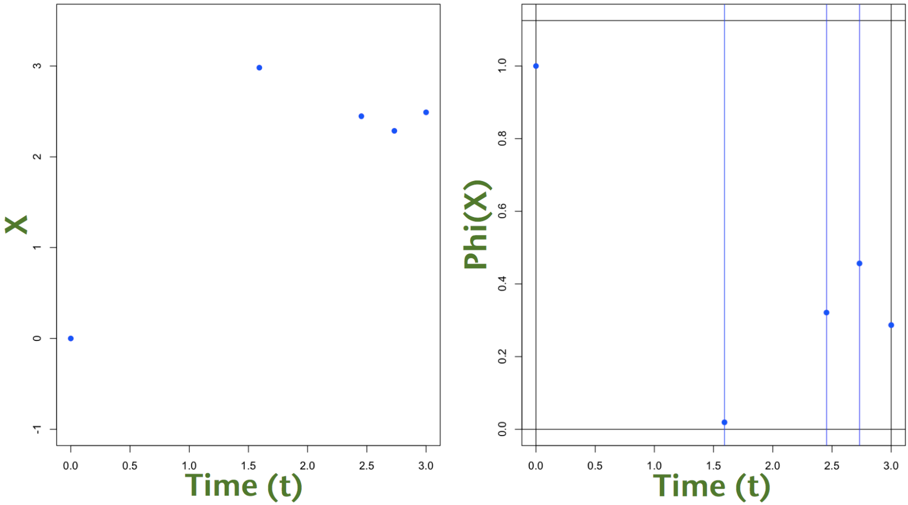
Exact Algorithm V



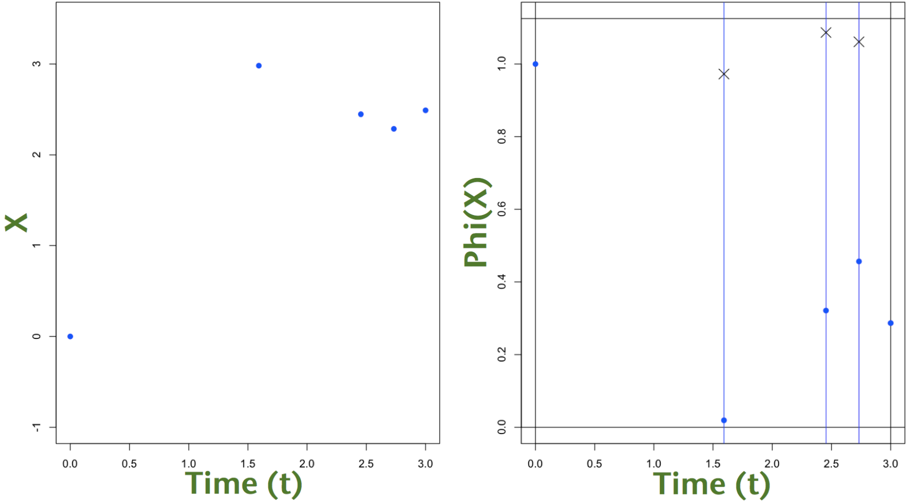
Exact Algorithm VI



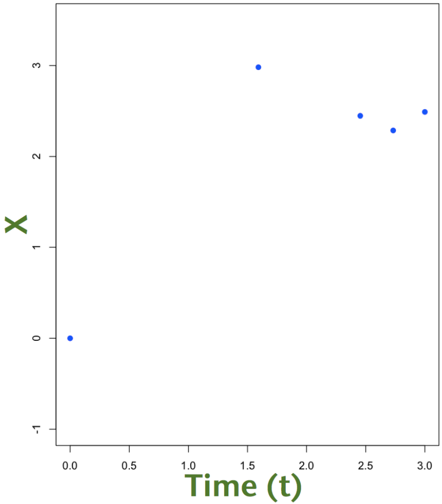
Exact Algorithm VII



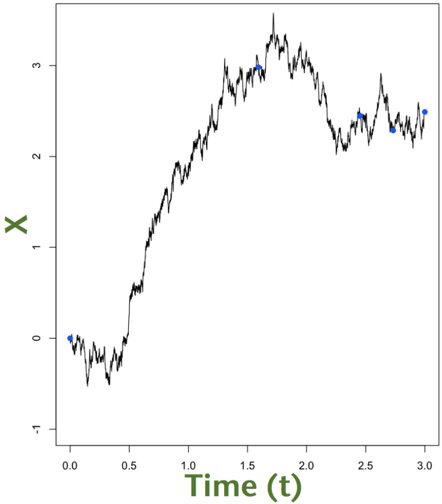
Exact Algorithm VIII



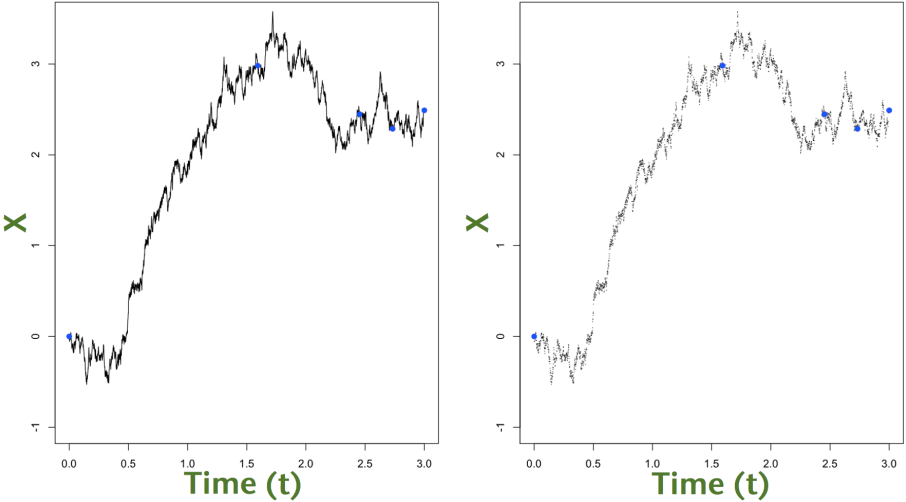
Exact Algorithm IX



Exact Algorithm X



Exact Algorithm XI



Random Weights I

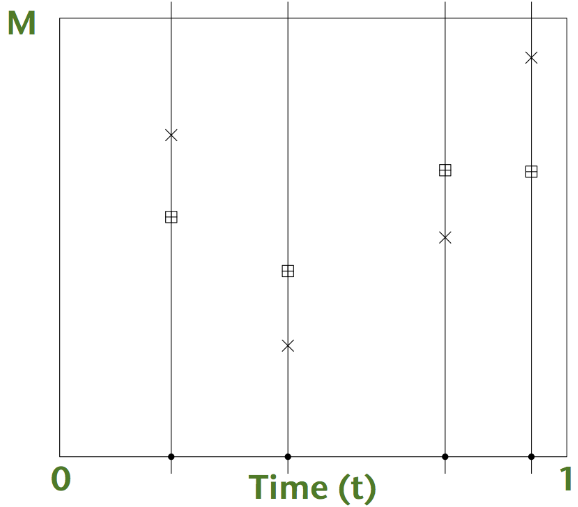
Brownian Bridge
- No Jumps
Dachuna-Castelle

$$p_{t-s}(x,y) = \underbrace{\mathcal{N}_{t-s}\,(y-x)\exp\{A(X_t) - A(X_s) - \ell(t-s)\}}_{h_{t-s}(x,y)} \underbrace{\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]}_{\text{weight}}$$

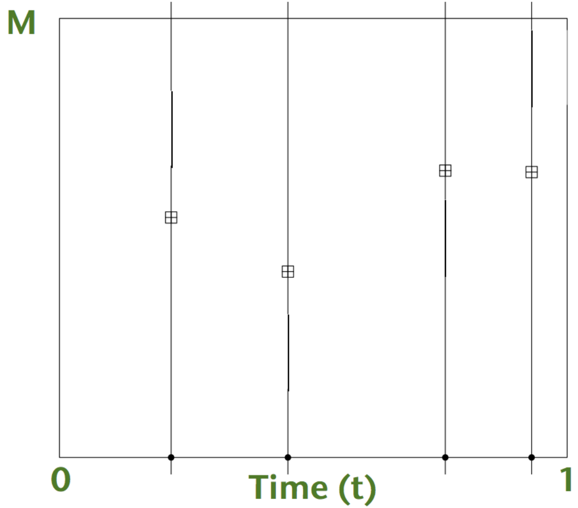
Transition Density

where, $\phi(X_u) := \dfrac{\alpha^2(X_u) + \alpha'(X_u)}{2} - \ell$

Random Weights II



Random Weights III



Poisson Estimator I

Brownian Bridge
- No Jumps
Dachuna-Castelle

$$p_{t-s}(x,y) = \underbrace{\mathcal{N}_{t-s}\,(y-x)\exp\{A(X_t)-A(X_s)-\ell(t-s)\}}_{h_{t-s}(x,y)}\ \underbrace{\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]}_{\text{weight}}$$

Transition Density

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Poisson Estimator

- **Key Idea:** Poisson Estimator - Simulate unbiased importance weight,

$$\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]$$

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- **What if** $\phi(X_u) \in [0, M]$**?**

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  - Broader class of Poisson Estimators

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  - Broader class of Poisson Estimators
    - Unbiased
    - Finite Variance (Generalised Poisson Estimators (FPR08))
    - Positivity (Wald Generalised Poisson Estimator (FPRS10))

Poisson Estimator

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  - Broader class of Poisson Estimators
    - Unbiased
    - Finite Variance (Generalised Poisson Estimators (FPR08))
    - Positivity (Wald Generalised Poisson Estimator (FPRS10))
  - Auxiliary Poisson Estimators (APES (PJR12b))
  - Generalised APES (GRAPES (PJR12b) - Jump Diffusions)

Particle Filter I

$\pi_\theta(x_t|x_{t-1}, y_t)$

Particle Filter II

$$\pi_\theta(x_t|x_{t-1}, y_t) = g_\theta(y_t|x_t) f_\theta(x_t|x_{t-1})$$

Particle Filter III

$$
\begin{aligned}
\pi_\theta(x_t|x_{t-1},y_t) &= g_\theta(y_t|x_t)f_\theta(x_t|x_{t-1}) \\
&= g_\theta(y_t|x_t)h_{t-(t-1)}(x_{t-1},x_t)\,\mathbb{E}_{\mathbb{W}^{x_{t-1},x_t}_{t-1,t}}\left[\exp\left\{-\int_{t-1}^{t}\phi(X_u)\,\mathrm{d}u\right\}\right]
\end{aligned}
$$

Particle Filter IV

$$\begin{aligned}\pi_\theta(x_t|x_{t-1},y_t) &= g_\theta(y_t|x_t)f_\theta(x_t|x_{t-1}) \\ &= \underbrace{g_\theta(y_t|x_t)h_{t-(t-1)}(x_{t-1},x_t)}_{\text{proposal}}\underbrace{\mathbb{E}_{\mathbb{W}^{x_{t-1},x_t}_{t-1,t}}\left[\exp\left\{-\int_{t-1}^{t}\phi(X_u)\,\mathrm{d}u\right\}\right]}_{\text{weight}}\end{aligned}$$

Jump Transition Density

Jump Transition Density I

$$\underset{\substack{\uparrow\\ \text{Transition Density}}}{p_{t-s}}(x,y) = \overset{\substack{\text{BB / JBB}\\ \downarrow}}{w_{t-s}}(x,y)\,\mathbb{E}_{\underset{\substack{\nwarrow\\ \text{BB / JBB}\\ \text{Measure}}}{\mathbb{W}^{x,y}_{s,t}}}\left[\overset{\substack{\text{BB / JBB}\\ \text{R-N Derivative}\\ \swarrow}}{\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}}(X)\right]$$

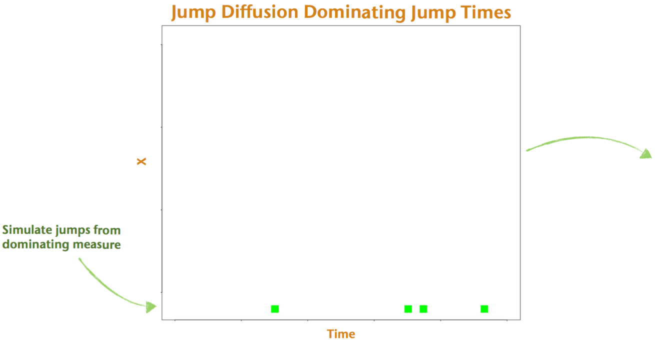
Jump Transition Density II

Jump Process

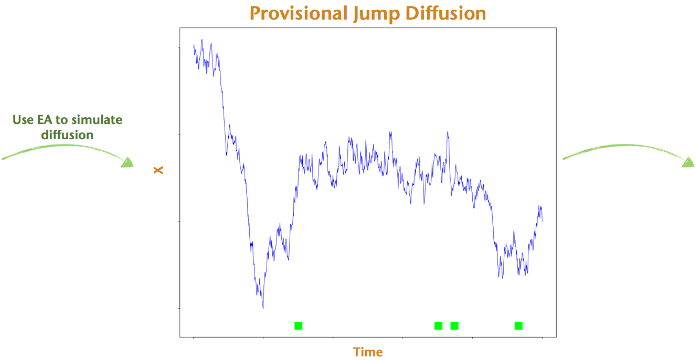
$$\underset{\uparrow}{p_{t-s}}(x,y) = \overset{\downarrow}{\mathbb{E}_{\mathbb{S}}}\mathbb{E}_{\mathbb{W}^{x,y}_{s,t}|\mathbb{S}}\left[w_{t-s}(x,y)\frac{\mathrm{d}\mathbb{Q}^{x}_{s}}{\mathrm{d}\mathbb{W}^{x}_{s}}(X) \middle| t_{J_1},\ldots,t_{J_M},\Delta X_{t_{J_1}},\ldots,\Delta X_{t_{J_M}}\right]$$

Transition Density

Jump Exact Algorithm I

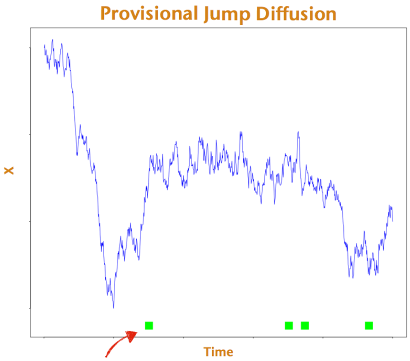


Jump Exact Algorithm II



Jump Exact Algorithm III

Sequentially evaluate diffusion at dominating jump times

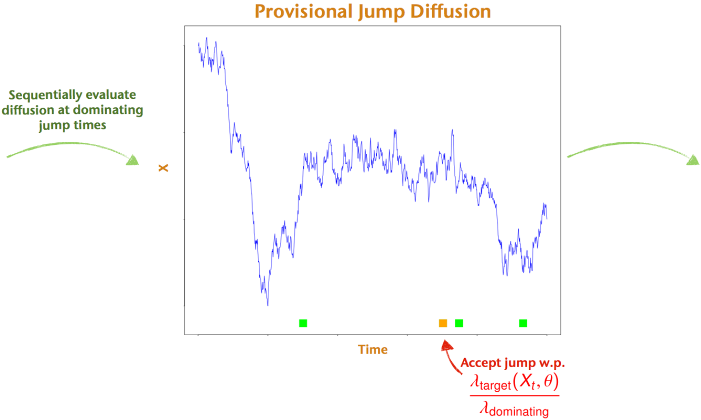


Provisional Jump Diffusion

Reject jump w.p.

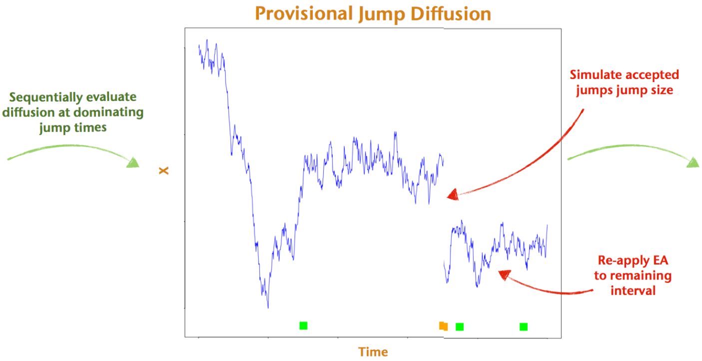
$$1 - \frac{\lambda_{\text{target}}(X_t, \theta)}{\lambda_{\text{dominating}}}$$

Jump Exact Algorithm IV



$$\frac{\lambda_{\text{target}}(X_t, \theta)}{\lambda_{\text{dominating}}}$$

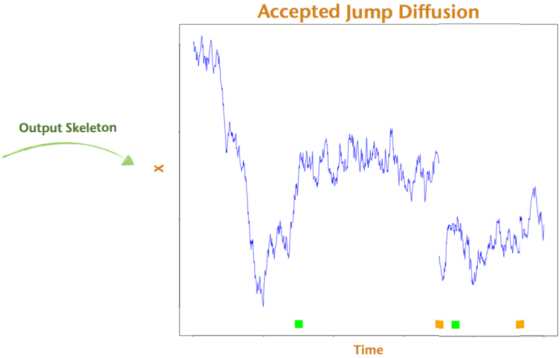
Jump Exact Algorithm V



Jump Exact Algorithm VI

Sequentially evaluate diffusion at dominating jump times

Jump Exact Algorithm VII



Example

Example: Particle Filtering for Jump Diffusions I

An Example. . .

- Exact Particle Filtering for Jump Diffusions In Practice!

Example: Particle Filtering for Jump Diffusions II

State Space Dynamics

$X_0 \sim N(0,5)$

$Y_t|(X_t = x_t) \sim N(x_t, 10)$

$t \in [0, 100]$

$\mathrm{d}X_t = \sin(X_t)\,\mathrm{d}t + \mathrm{d}B_t + \mathrm{d}J_t$

$\lambda(X_t) = \cos^2(X_t)$

$\mu(X_t) \sim N(\sin(X_t), 1)$

Example: Particle Filtering for Jump Diffusions III

State Space Dynamics

$$X_0 \sim N(0,5)$$
$$Y_t|(X_t = x_t) \sim N(x_t, 10)$$
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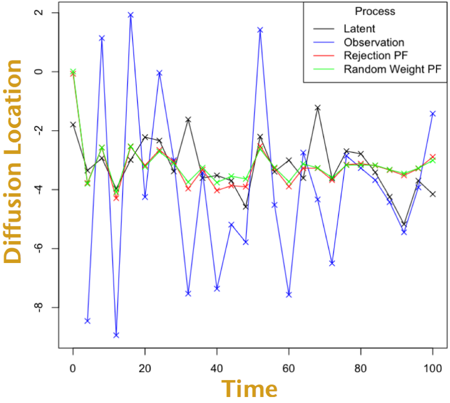
$$\lambda(X_t) = \cos^2(X_t)$$
$$\mu(X_t) \sim N(\sin(X_t), 1)$$

Process and Particle Filter Estimates



Example: Particle Filtering for Jump Diffusions IV

State Space Dynamics

$X_0 \sim N(0,5)$

$Y_t|(X_t = x_t) \sim N(x_t, 10)$

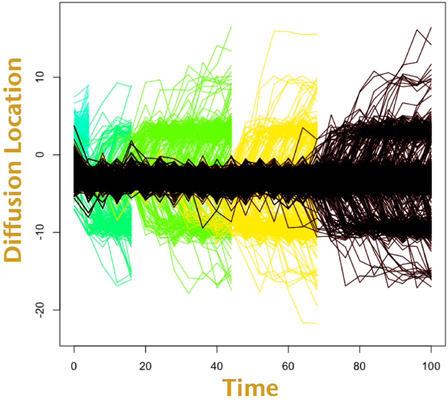
$t \in [0, 100]$

$dX_t = \sin(X_t)\,dt + dB_t + dJ_t$

$\lambda(X_t) = \cos^2(X_t)$

$\mu(X_t) \sim N(\sin(X_t), 1)$

Exact Rejection PF - Particle Paths



Example: Particle Filtering for Jump Diffusions V

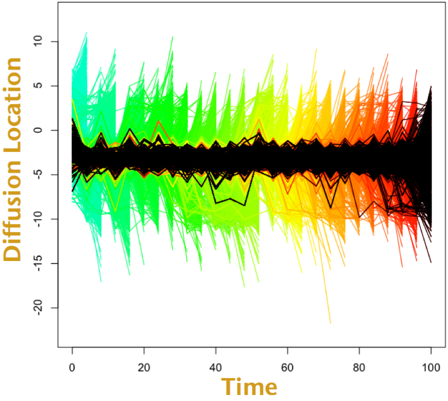
State Space Dynamics

$X_0 \sim N(0,5)$ $\quad$ $\mathrm{d}X_t = \sin(X_t)\,\mathrm{d}t + \mathrm{d}B_t + \mathrm{d}J_t$

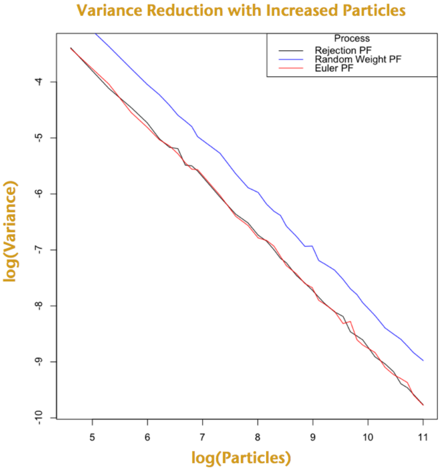
$Y_t|(X_t = x_t) \sim N(x_t, 10)$ $\quad$ $\lambda(X_t) = \cos^2(X_t)$

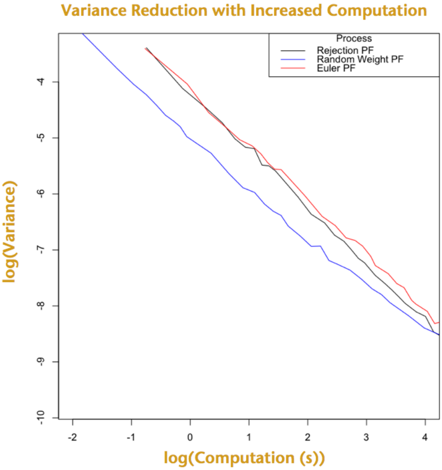
$t \in [0, 100]$ $\quad$ $\mu(X_t) \sim N(\sin(X_t), 1)$

Exact Random Weight PF - Particle Paths



Example: Particle Filtering for Jump Diffusions VI


Variance Reduction with Increased Particles


Variance Reduction with Increased Computation

Other (Related) Work

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Current Work

- **Poisson Estimator**
  - As discussed…
  - $\min_{u\in[s,t]} \alpha^2(X_u) + \alpha'(X_u)$

Other (Related) Work

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- **Poisson Estimator**
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  - $\min_{u \in [s,t]} \alpha^2(X_u) + \alpha'(X_u)$
- **Exact Algorithm**
  - Alternative to EA3 (EA4 - ‘Mondrian’ EA)
  - EA3 Implementation
  - Jump EA

Other (Related) Work

Current Work

- **Poisson Estimator**
  - As discussed...
  - $\min_{u \in [s,t]} \alpha^2(X_u) + \alpha'(X_u)$
- **Exact Algorithm**
  - Alternative to EA3 (EA4 - 'Mondrian' EA)
  - EA3 Implementation
  - Jump EA
- **ϵ-Strong Simulation (BPR12)**
  - Initialisation
  - Various Sampling Steps
  - Extensions

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[PJR12b] M. Pollock, A. M. Johansen, and G. O. Roberts. Exact Particle Filters. Technical report, CRISM, Department of Statistics, University of Warwick., 2012.

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