ϵ-Strong Simulation of Diffusions

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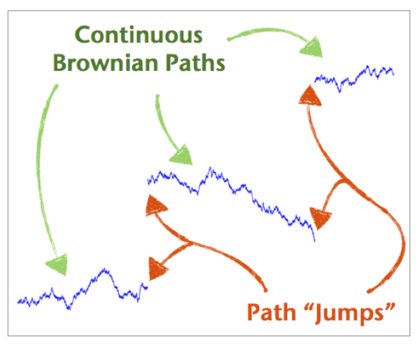
1 - Problem Outline

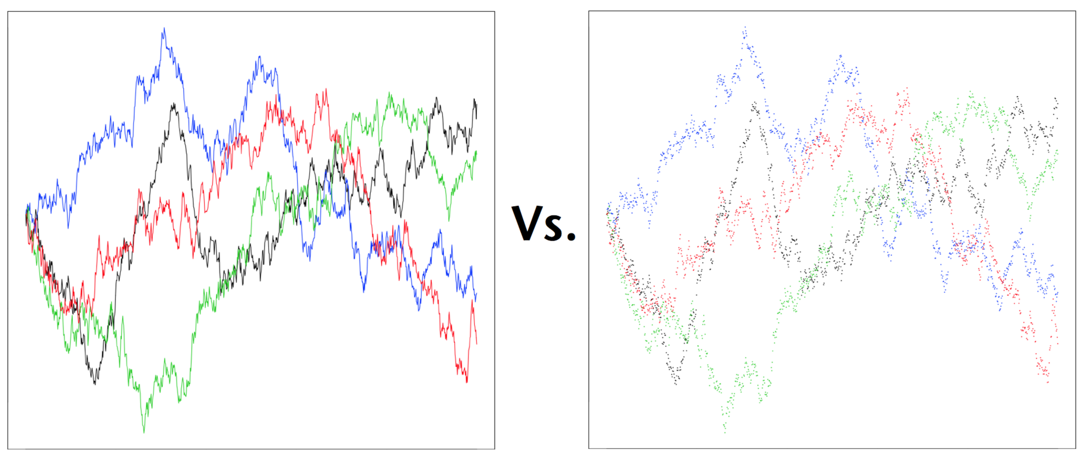
1.1 - The Goal...??? Simulate upper and lower convergent dominating processes ($X^{\downarrow}$ and $X^{\uparrow}$) which enfold almost surely diffusion sample paths over some finite interval.

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \mathrm{d}J_t\big[\lambda(X_t); \nu(X_t)\big]$$

Jump Diffusion; Instantaneous Mean; Instantaneous Volatility; Standard Brownian Motion; Jump Rate Intensity; Jump Size Density; Poisson Jump Process



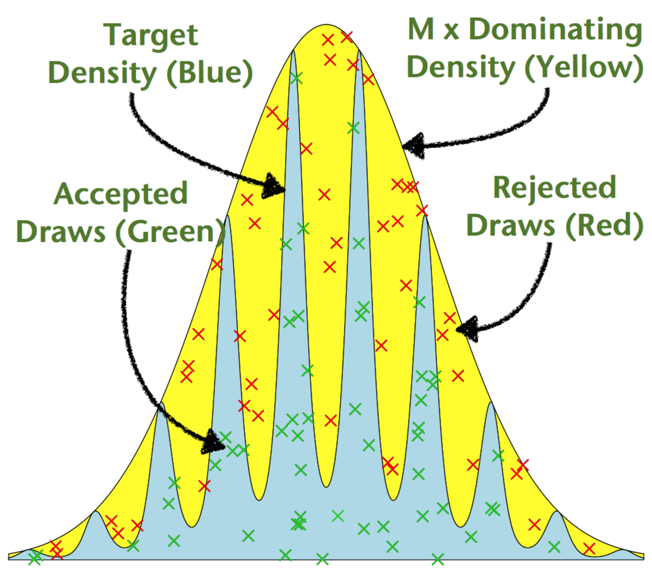
1.2 - Key Point... Diffusion sample paths can only be simulated at a finite number of points. Between any pair of points the behaviour of the sample path is unknown. . .



1.3 - Applications... Monte Carlo Integration, Option Pricing, Simulating Hitting Times, Extrema etc. . .

2 - Key Ideas

2.1 - Rejection Sampling



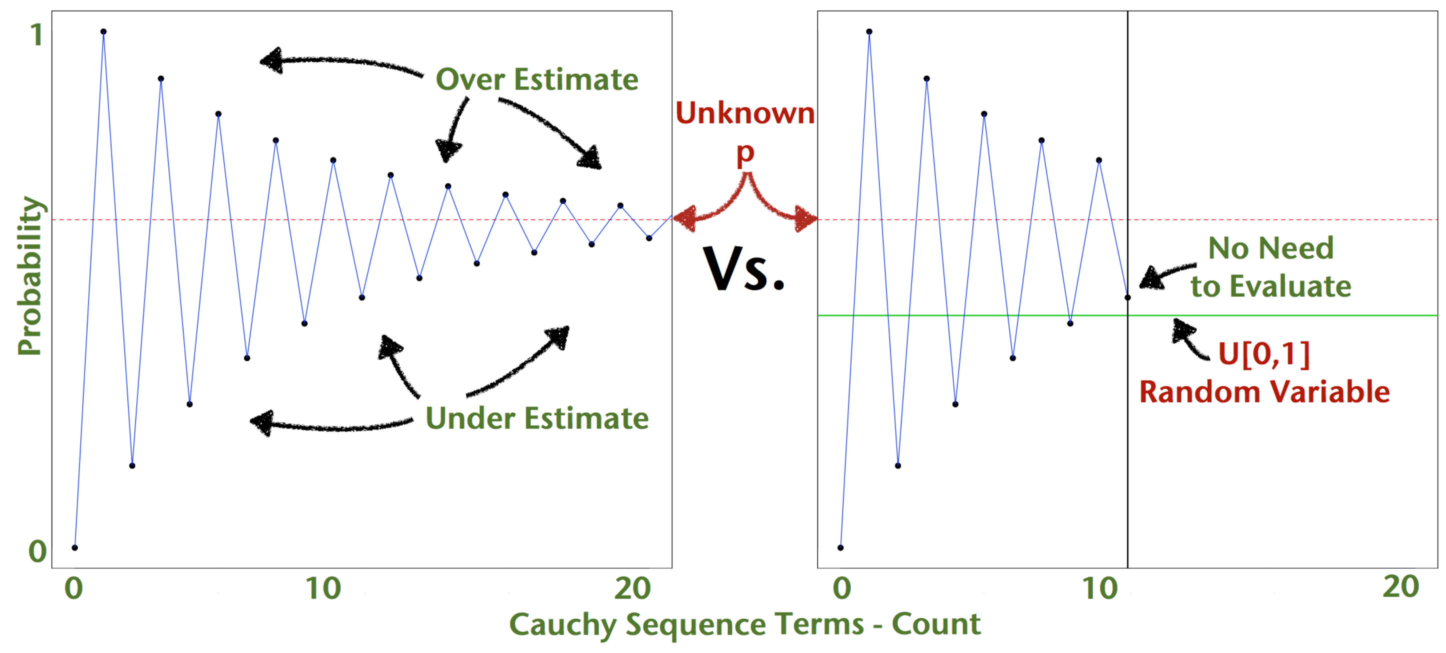
2.2 - Expectations of (+)ve Random Variables

- **Goal** - Evaluate $\mathbb{E}(P)$; $P \in [0,1]$ RV; $u \sim U[0,1]$
- **Indicator** - **If** $u \leq P$ **then** $C_p = 1$ **else** $C_p = 0$

$$\mathbb{P}(C_p = 1) = \mathbb{E}\big(\mathbb{1}(u \leq P)\big) = \mathbb{E}\Big(\mathbb{E}\big(\mathbb{1}(u \leq P)\big|P\big)\Big) = \mathbb{E}\Big(\mathbb{P}\big(C_p = 1\big|P\big)\Big) = \mathbb{E}(P)$$

- **Simulate C_p to unbiasedly evaluate $\mathbb{E}(P)$!!**

2.3 - Retrospective Inversion Sampling



2.4 - Exact Algorithm

- The transition density of a diffusion (typically) can not be evaluated analytically.

$$p_{t-s}(y|x) = h_{t-s}(x,y)\cdot \mathbb{E}_{\mathbb{W}^{x,y}_{[s,t]}}\left[\exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}\right]$$

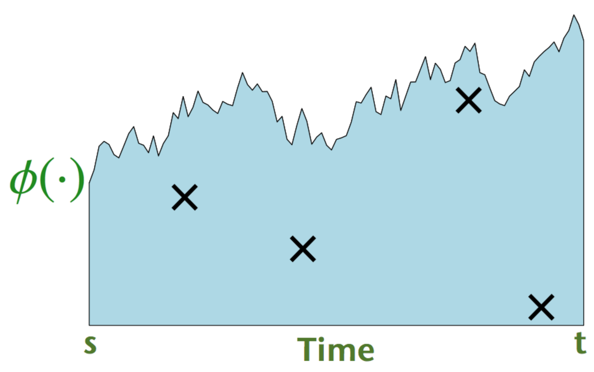
Transition Density; Elapsed Time; Brownian Bridge; Function of Diffusion Drift; $X_s = x$ $X_t = y$; 2.2

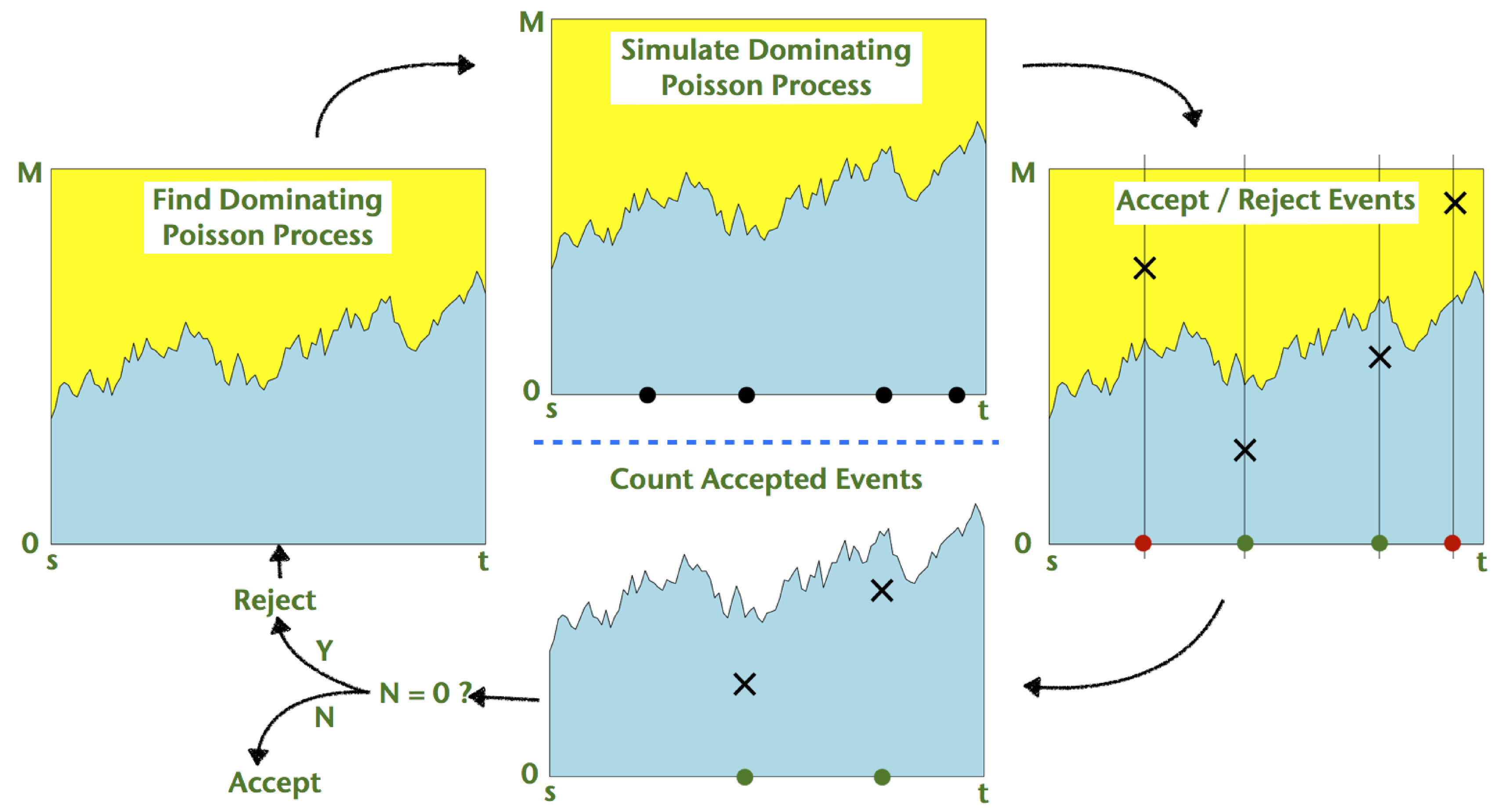
$$\phi(\cdot) \in [0, M] \Rightarrow P := \exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\} \in [0,1] \Rightarrow \mathbb{E}_{\mathbb{W}^{x,y}_{[s,t]}}\big[P\big] = P$$

- Now, consider a Poisson process with instantaneous rate $\phi(\cdot)$ on the interval $[s,t]$,

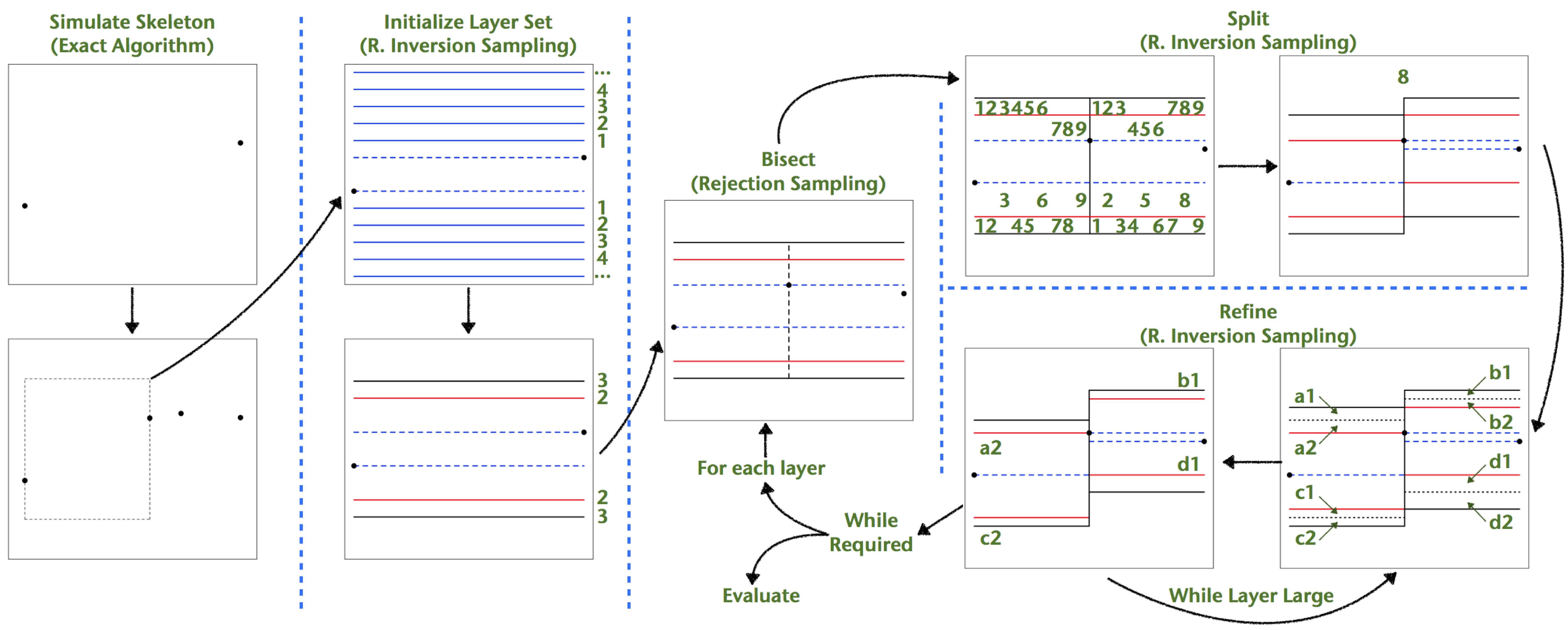
$$\mathbb{P}(N = 0) = \exp\left\{-\int_s^t \phi(X_u)\,\mathrm{d}u\right\}$$

Counting Process; Rate Parameter; Number of Events

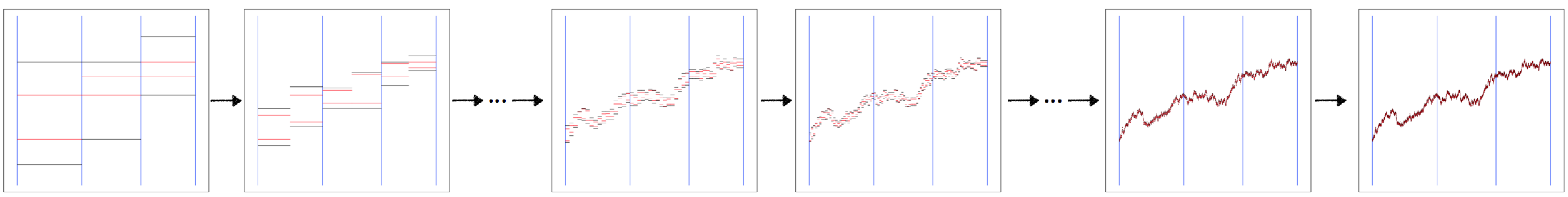




3 - Algorithm



4 - Example



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