Questions from the Lévy processes reading group 2

1. Determine $a$, $Q$ and $\pi$ in the Lévy-Khintchine formula for Gaussian and Poisson random variables.

2. Suppose $(\xi_n)_{n \geq 1}$ are i.i.d. with law $\nu$ on $\mathbb{R} \setminus \{0\}$, independent of a Poisson process $N = (N_t)_{t \geq 0}$ of rate $c$. Check that if we define

$$e(t) := \begin{cases} 
\xi_n & \text{if } N_{t-} < n = N_t, \\
0 & \text{otherwise},
\end{cases}$$

then $e = (e(t))_{t \geq 0}$ is a Poisson point process with characteristic measure $c\nu$ (in the sense of Section 0.5 of the book).

3. In the setting of question 2, check that the compound Poisson process

$$\sum_{n=1}^{N_t} \xi_n = \sum_{s \leq t} e(s)$$

is a Lévy process with characteristic exponent

$$\psi(\lambda) = c \int_{\mathbb{R}^d} (1 - e^{i\langle \lambda, x \rangle}) \nu(dx).$$

4. Continue in the setting of question 2, but now suppose that the finite measure $\pi := c\nu$ is supported only upon $[-1, 1] \setminus \{0\}$. Check that $M = (M_t)_{t \geq 0}$, where

$$M_t := \sum_{s \leq t} e(s) - t \int_{[-1,1] \setminus \{0\}} x \pi(dx),$$

is a martingale. Show that its variance at time $t$ is given by

$$t \int_{[-1,1] \setminus \{0\}} x^2 \pi(dx).$$

5. In the setting of question 4, use Doob’s $L^2$ (sub-)martingale inequality to confirm the result from the reading group that

$$\mathbb{E} \left( \sup_{s \leq t} \left| \sum_{r \leq s} c(r) - s \int_{[-1,1] \setminus \{0\}} x \pi(dx) \right| \right)^2 \leq 4t \int_{[-1,1] \setminus \{0\}} x^2 \pi(dx).$$